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AN

INTRODUCTION

TO

ALGEBRA,

WITH

NOTES AND OBSERVATIONS;

DESIGNED FOR THE

USE OF SCHOOLS, AND OTHER PLACES OF PUBLIC EDUCATION.

By JOHN BONNYCASTLE,

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THE THIRTEENTH EDITION.

——Ingenuas didicisse fideliter artes
Emollit mores, nec sinit esse feros. *Ovid.*

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PREFACE.

THE powers of the mind, like those of the body, are increased by frequent exertion; application and industry supply the place of genius and invention; and even the creative faculty itself may be strengthened and improved by use and perseverance. Uncultivated nature is uniformly rude and imbecile, it being by imitation alone that we at first acquire knowledge, and the means of extending its bounds. A just and perfect acquaintance with the simple elements of science, is a necessary step towards our future progress and advancement; and this, assisted by laborious investigation and habitual inquiry, will constantly lead to eminence and perfection.

Books of rudiments, therefore, concisely written, well digested, and methodically arranged, are treasures of inestimable value; and too many attempts cannot be made to render them perfect and complete. When the first principles of any art or science are firmly fixed and rooted in the mind, their application soon becomes easy, pleasant and obvious: the understanding is delighted and enlarged; we conceive clearly, reason distinctly, and form just and satisfactory conclusions. But, on the contrary, when the mind, instead of reposing on the stability of truth and received principles, is wandering in doubt and uncertainty, our ideas will necessarily be confused and obscure; and every step we take must be attended with fresh difficulties and endless perplexity.

That the grounds, or fundamental parts, of every science, are dull and unentertaining, is a complaint universally made, and a truth not to be denied; but then, what is obtained with difficulty is usually remem-

bered with ease; and what is purchased with pain is often possessed with pleasure. The seeds of knowledge are sown in every soil, but it is by proper culture alone that they are cherished and brought to maturity. A few years of early and assiduous application, never fails to procure us the reward of our industry; and who that knows the pleasures and advantages which the sciences afford, would think his time, in this case, mis-spent, or his labours useless? Riches and honours are the gifts of fortune, casually bestowed, or hereditarily received, and are frequently abused by their possessors; but the superiority of wisdom and knowledge is a pre-eminence of merit, which originates with the man, and is the noblest of all distinctions.

Nature, bountiful and wise in all things, has provided us with an infinite variety of scenes, both for our instruction and entertainment; and, like a kind and indulgent parent, admits all her children to an equal participation of her blessings. But, as the modes, situations, and circumstances of life are various, so accident, habit, and education, have each their predominating influence, and give to every mind its particular bias. Where examples of excellence are wanting, the attempts to attain it are but few; but eminence excites attention, and produces imitation. To raise the curiosity, and to awaken the listless and dormant powers of younger minds, we have only to point out to them a valuable acquisition, and the means of obtaining it; the active principles are immediately put into motion, and the certainty of the conquest is ensured from a determination to conquer.

But of all the sciences which serve to call forth this spirit of enterprise and inquiry, there are none more eminently useful than the Mathematics. By an early attachment to these elegant and sublime studies, we acquire a habit of reasoning, and an elevation of thought,

which fixes the mind, and prepares it for every other pursuit. From a few simple axioms, and evident principles, we proceed gradually to the most general propositions, and remote analogies: deducing one truth from another, in a chain of argument well connected and logically pursued; which brings us at last, in the most satisfactory manner, to the conclusion, and serves as a general direction in all our inquiries after truth.

And it is not only in this respect that mathematical learning is so highly valuable; it is, likewise, equally estimable for its practical utility. Almost all the works of art and devices of man, have a dependence upon its principles, and are indebted to it for their origin and perfection. The cultivation of these admirable sciences is, therefore, a thing of the utmost importance, and ought to be considered as a principal part of every liberal and well-regulated plan of education. They are the guide of our youth, the perfection of our reason, and the foundation of every great and noble undertaking.

From these considerations, I have been induced to compose an introductory course of mathematical science; and, from the kind encouragement which I have hitherto received, am not without hopes of a continuance of the same candour and approbation. Considerable practice as a teacher, and a long attention to the difficulties and obstructions which retard the progress of learners in general, have enabled me to accommodate myself the more easily to their capacities and understandings. And as an earnest desire of promoting and diffusing useful knowledge is the chief motive for this undertaking, so no pains or attention shall be wanting to make it as complete and perfect as possible.

The subject of the present performance is ALGEBRA; which is one of the most important and useful branches of those sciences, and may be justly considered as the key to all the rest. Geometry delights us by the sim-

plicity of its principles, and the elegance of its demonstrations: Arithmetic is confined in its object, and partial in its application; but Algebra, or the analytic art, is general and comprehensive, and may be applied with success in all cases where truth is to be obtained and proper data can be established.

To trace this science to its birth, and to point out the various alterations and improvements it has undergone in its progress, would far exceed the limits of a preface*. It will be sufficient to observe that the invention is of great antiquity, and has challenged the praise and admiration of all ages. DIOPHANTUS, a Greek mathematician, of Alexandria in Egypt, who flourished in or about the fourth century after CHRIST, appears to have been the first, among the ancients, who applied it to the solution of indeterminate, or unlimited problems; but it is to the moderns that we are principally indebted for the most curious refinements of the art, and its great and extensive usefulness in every abstruse and difficult inquiry. NEWTON, MACLAURIN, SAUNDERSON, SIMPSON, and EMERSON, among our own countrymen, and CLAIRAUT, EULER, LAGRANGE, and LACROIX, on the continent, are those who have particularly excelled in this respect; and it is to their works that I would refer the young student, as the patterns of elegance and perfection.

The following compendium is formed entirely upon the model of those writers, and is intended as a useful

* Those who are desirous of a knowledge of this kind, may consult the Introduction to my *Treatise on Algebra*, 2d Edition, 2 Vols. 8vo. 1820, where they will find a regular historical detail of the rise and progress of the science, from its first rude beginnings to the present time; together with a variety of other particulars, relating to the theoretical and practical part of the subject, which are there more fully explained and developed, than could have been done in a compendium like the present.

and necessary introduction to them. Almost every subject, which belongs to pure Algebra, is concisely and distinctly treated of; and no pains have been spared to make the whole as easy and intelligible as possible. A great number of elementary books have already been written upon this subject; but there are none, which I have yet seen, but what appear to me to be extremely defective. Besides being totally unfit for the purpose of teaching, they are generally calculated to vitiate the taste, and mislead the judgment. A tedious and inelegant method prevails through the whole, so that the beauty of the science is generally destroyed by the clumsy and awkward manner in which it is treated; and the learner, when he is afterwards introduced to some of our best writers, is obliged, in a great measure, to unlearn and forget every thing which he has been at so much pains in acquiring.

There is a certain taste and elegance in the sciences, as well as in every branch of polite literature, which is only to be obtained from the best authors, and a judicious use of their instructions. To direct the student in his choice of books, and to prepare him properly for the advantages he may receive from them, is therefore the business of every writer who engages in the humble, but useful task of a preliminary tutor. This information I have been careful to give, in every part of the present performance, where it appeared to be in the least necessary; and, though the nature and confined limits of my plan admitted not of diffuse observations, or a formal enumeration of particulars, it is presumed nothing of real use and importance has been omitted. My principal object was to consult the ease, satisfaction, and accommodation of the learner; and the favourable reception the work has met with from the public, has afforded me the gratification of believing that my labours have not been unsuccessfully employed.

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TO

THE TENTH EDITION.

The present performance having passed through a number of editions since the time of its first publication, without any material alterations having been made, either with respect to its original plan, or the manner in which it was executed, I have been induced, from the flattering approbation it has constantly received, to undertake an entire revision of the work; and, by availing myself of the improvements that have been subsequently made in the science, to render it still more deserving the public favour.

In its present state, it may be considered as a concise abridgment of the most practical and useful parts of my larger work on this subject, mentioned in the Preface; from which, except in certain cases, where a

different mode of proceeding appeared to be necessary, it has been chiefly compiled: great care having been taken, at the same time, to adapt it, as much as possible, to the wants of learners, and the general purposes of instruction, agreeably to the design with which it was first written.

With this view, as well as in compliance with the wishes of several intelligent teachers, I have also been led to subjoin to it, by way of an Appendix, a small Tract on the application of Algebra to the solution of Geometrical Problems; which, it is hoped, will prove acceptable to such classes of students as may not have an opportunity of consulting more voluminous and expensive works on this interesting branch of the science.

JOHN BONNYCASTLE.

ROYAL MILITARY ACADEMY,
WOOLWICH,
October 22, 1815.

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TO

THE THIRTEENTH EDITION.

CONSIDERABLE improvements having lately been made in the Solution of Equations by Approximation, a subject of great importance in Algebra, I have been induced to add an Addenda to the present Edition of this work, containing an entirely new method for that purpose; which I trust will be found, in many respects, more convenient than any hitherto published.

CHARLES BONNYCASTLE.

CHATHAM,

July 19, 1824.

CONTENTS.

	Page
DEFINITIONS	1
Addition	8
Subtraction	12
Multiplication	13
Division	18
Algebraic Fractions	24
Involution, or the raising of Powers	40
Evolution, or the Extraction of Roots	44
Of irrational Quantities, or Surds	50
Of arithmetical Proportion and Progression	72
Of geometrical Proportion and Progression	76
Of Equations	81
Of the Resolution of simple Equations	83
Miscellaneous Questions	98
Of quadratic Equations	103
Questions producing quadratic Equations	116
Of cubic Equations	123
Of the solution of cubic Equations	125
Of the Resolution of biquadratic Equations	130
To find the roots of Equations by Approximation	133
To find the roots of Exponential Equations	140

Page
Of the Binomial Theorem 143
Of the Indeterminate Analysis 150
Of the Diophantine Analysis 162
Of the Summation and Interpolation of Series . . 173
Of Logarithms 200
Multiplication by Logarithms 213
Division by Logarithms 216
The Rule of Three, by Logarithms 218
Involution, by Logarithms 221
Evolution, by Logarithms 223
A Collection of Miscellaneous Questions 226

APPENDIX, on the application of Algebra to Geometry 233

ADDENDA, on the Solution of Equations by Approximation 261

ALGEBRA.

Algebra is the science which treats of a general method of performing calculations, and resolving mathematical problems, by means of the letters of the alphabet.

Its leading rules are the same as those of arithmetic; and the operations to be performed are denoted by the following characters:

$+$ *plus*, or *more*, the sign of addition; signifying that the quantities between which it is placed are to be added together.

Thus, $a+b$ shows that the number, or quantity, represented by b, is to be added to that represented by a; and is read a plus b.

$-$ *minus*, or *less*, the sign of subtraction; signifying that the latter of the two quantities between which it is placed is to be taken from the former.

Thus, $a-b$ shows that the number, or quantity, represented by b is to be taken from that represented by a; and is read a minus b.

Also, $a \sim b$ represents the difference of the two quantities a and b, when it is not known which of them is the greater.

$\times$ *into*, the sign of multiplication; signifying that the quantities between which it is placed are to be multiplied together.

Thus, $a \times b$ shows that the number, or quantity, represented by a is to be multiplied by that represented by b; and is read a into b.

The multiplication of simple quantities is also frequently denoted by a point, or by joining the letters together in the form of a word.

Thus, $a \times b$, $a.b$, and ab, all signify the product of a and b: also, $3 \times a$, or $3a$, is the product of 3 and a: and is read 3 times a.

$\div$ *by*, the sign of division; signifying that the former of the two quantities between which it is placed is to be divided by the latter.

Thus, $a \div b$ shows that the number, or quantity, represented by a is to be divided by that represented by b; and is read a by b, or a divided by b.

Division is also frequently denoted by placing one of the two quantities over the other, in the form of a fraction.

Thus, $b \div a$ and $\dfrac{b}{a}$ both signify the quotient of b divided by a; and $\dfrac{a-b}{a+c}$ signifies that $a-b$ is to divided by $a+c$.

$=$ *equal to*, the sign of equality; signifying that the quantities between which it is placed are equal to each other.

Thus, $x=a+b$ shows that the quantity denoted by x is equal to the sum of the numbers, or quantities, a and b; and is read x equal to a plus b.

$\equiv$ *identical to*, or the sign of equivalence; signifying that the expressions between which it is placed are equal for all values of the letters of which they are composed.

Thus, $\frac{1}{2}(a+x)+\frac{1}{2}(a-x)\equiv a$; $\frac{1}{2}(a+x)-\frac{1}{2}(a-x)\equiv x$; and $(x+a)\times(x-a)\equiv x^2-a^2$, whatever numeral values may be given to the quantities represented by x and a.

$>$ *greater than*, the sign of majority; signifying that the former of the two quantities between which it is placed is greater than the latter.

Thus $a>b$ shows that the number, or quantity, represented by a is greater than that represented by b; and is read a greater than b.

$<$ *less than*, the sign of minority; signifying that the

former of the two quantities between which it is placed is less than the latter.

Thus, $a \sqsubset b$ shows that the number or quantity, represented by a is less than that represented by b; and is read a less than b.

: *as*, or *to*, and :: *so is*, the signs of an equality of ratios; signifying that the quantities between which they are placed are proportional.

Thus, $a : b :: c : d$ denotes that a has the same ratio to b that c has to d, or that a, b, c, d, are proportionals; and is read as a is to b so is c to d, or a is to b as c is to d.

$\surd$ *the radical sign*, signifying that the quantity before which it is placed is to have some root of it extracted.

Thus, $\sqrt{a}$ is the square root of a; $\sqrt[3]{a}$ is the cube root of a; $\sqrt[4]{a}$ is the fourth root of a; and so on.

The roots of quantities, are also frequently represented by figures placed at the right hand corner of them, in the form of a fraction.

Thus ,$a^{\frac{1}{2}}$ is the square root of a; $a^{\frac{1}{3}}$ is the cube root of a; $a^{\frac{1}{4}}$ is the fourth root of a; and $a^{\frac{1}{n}}$, or $\sqrt[n]{a}$, is the nth root of a, or a root denoted by any number n.

In like manner, a^2 is the square of a; a^3 is the cube of a; a^4 is the fourth power of a; and a^m is the mth power of a, or any power denoted by the number m.

∞ is the sign of infinity, signifying that the quantity standing before it is of an unlimited value, or greater than any quantity that can be assigned.

The coefficient of a quantity is the number or letter prefixed to it; being that which shows how often the other is to be taken.

Thus, in the quantities $3b$, $-\frac{2}{3}b$. 3 and $-\frac{2}{3}$ are the coefficients of b; and a is the coefficient of x in the quantity ax.

A quantity without any coefficient prefixed to it is supposed to have 1 or unity; and when a quantity has no sign before it, + is always understood.

Thus, a is the same as $+a$, or $+1a$; and $-a$ is the same as $-1a$.

A term is any part or member of a compound quantity, which is separated from the rest by the signs + or —.

Thus, a and b are the terms of $a+b$; and $3a$, $-2b$, and $+5cd$, are the terms of $3a-2b+5cd$.

In like manner, the terms of a product, fraction, or proportion, are the several parts or quantities of which they are composed.

Thus, a and b are the terms of ab, or of $\frac{a}{b}$; and a, b, c, d, are the terms of the proportion $a : b :: c : d$.

Factors are the numbers, or quantities, from the multiplication of which other numbers, or quantities, are produced.

Thus, a and b are the factors of ab; also, 2, a, and b^2, are the factors of $2ab^2$; and $a+x$ and $a-x$ are the factors of the product $(a+x)\times(a-x)$.

Like quantities, are those which consist of the same letters or combinations of letters, or that differ only in their coefficients: as a and $3a$, or $5ab$ and $7ab$, or $2a^2b$ and $9a^2b$.

Unlike quantities, are those which consist of different letters, or combinations of letters; as a and b, or $3a$ and a^2, or $5ab^2$ and $7a^2b$.

Given quantities, are such as have known values, and are generally represented by some of the first letters of the alphabet; as a, b, c, d, *&c.*

Unknown quantities, are such as have no fixed or determinate values; and are usually represented by some of the final letters of the alphabet; as x, y, z.

Simple quantities, are those which consist of one term only; as $3a$, $5ab$, $-8a^2b$, *&c.*

Compound quantities, are those which consist of several terms; as $2a+b$, or $3a-2c$, or $a+2b-3c$, &c.

Positive, or affirmative quantities, are those that are to be added; or such as stand simply by themselves, or have the sign $+$ prefixed to them: as a, or $+a$, or $+3ab$, &c.

Negative quantities, are those that are to be subtracted; or such as have the sign $-$ prefixed to them; as $-a$, or $-3ab$, or $-7ab^2$, &c.

Like signs, are such as are all positive, or all negative; as $+$ and $+$, or $-$, and $-$.

Unlike signs, are when some are positive and others negative; as $+$ and $-$, or $-$ and $+$.

A monomial, is a quantity consisting of one term; as a, $2b$, $-3a^2b$, &c., being the same as a simple quantity, or one that stands by itself, without any connexion with others.

A binomial, is a quantity consisting of two terms; as $a+b$, or $a-b$; the latter of which is, also, sometimes called a residual quantity.

A trinomial, is a quantity, consisting of three terms, as $a+2b-3c$; a quadrinomial of four, as $a-2b+3c-d$, and a polynomial, or multinomial, is that which has many terms.

The power of a quantity, is its square, cube, biquadrate, &c.; called also its second, third, fourth power, &c.; as a^2, a^3, a^4, &c.

The index, or exponent of a quantity, is the number which denotes its power or root.

Thus, -1 is the index, or exponent, of a^{-1}, 2 is the index of a^2; $\frac{1}{2}$ of $a^{\frac{1}{2}}$ or $\sqrt{a}$; and m and $\frac{1}{n}$ of a^m and $a^{\frac{1}{n}}$

When a quantity appears without any index, or exponent, it is always understood to have unity, or 1.

Thus, a is the same as a^1, and $2x$ is the same as $2x^1$; the 1, in such cases, being usually omitted.

A rational number, or quantity, is that which can be

expressed in finite terms, or without any radical sign, or fractional index; as a, or $\frac{2}{3}a$, or $5a^2$, &c.

An irrational quantity, or surd, is that which has no exact root, or which can only be expressed by means of the radical sign, or a fractional index; as $\sqrt{2}$ or $2^{\frac{1}{2}}$, $\sqrt[3]{a^2}$ or $a^{\frac{2}{3}}$ &c.

A square or cube number, &c., is that which has an exact square or cube root, &c.

Thus, 4 and $\frac{9}{16}a^2$ are square numbers; and 64 and $\frac{8}{27}a^3$ are cube numbers, &c.

A measure, or divisor, of any quantity, is that which is contained in it some exact number of times.

Thus, 3 is a measure, or divisor, of 6, $7a$ is a measure of $35a$, and $9ab$ of $27a^2b^2$.

A composite number, or quantity, is that which is produced by the multiplication of two or more terms or factors.

Thus, 6 is a composite number, formed of the factors 2 and 3, or 2×3; and $3abc$ is a composite quantity, the factors of which are 3, a, b, c.

Commensurable numbers, or quantities, are such as can be each divided by the same quantity, without leaving a remainder.

Thus, 6 and 8, $2\sqrt{2}$, and $3\sqrt{2}$, $5a^2b$ and $7ab^2$, are commensurable quantities; the common divisors being 2, $\sqrt{2}$, and ab.

Incommensurable numbers, or quantities, are such as have no common measure, or divisor, except unity.

Thus, 2 and 7, 5 and 8, $\sqrt{2}$, and $\sqrt{3}$, and $a+b$ and a^2+b^2, are incommensurable quantities.

Also, when two numbers have no common measure, or divisor, except unity, they are said to be prime to each other; as is the case with the numbers 2 and 7, or 5 and 8 abovementioned.

A multiple of any quantity, is that which is some exact number of times that quantity.

Thus, 12 is a multiple of 4, $15a$ is a multiple of $3a$, and $20a^2b^2$ of $5ab$.

The reciprocal of any quantity, is that quantity inverted, or unity divided by it.

Thus, the reciprocal of a, or $\frac{a}{1}$, is $\frac{1}{a}$; and the reciprocal of $\frac{a}{b}$ is $\frac{b}{a}$.

A function of one or more quantities, is an expression into which those quantities enter, in any manner whatever, either combined, or not, with known quantities.

Thus, $a-2x$, $ax+3x^2$, $2x-a(a^2-x^2)^{\frac{1}{2}}$, ax^m, a^x, &c. are functions of x; and $axy\times bx^2$, $ay+x(ax^2-by^3)^{\frac{1}{2}}$, &c. are functions of x and y.

A vinculum, is a bar ——, or parenthesis ( ), made use of to collect several quantities into one.

Thus, $\overline{a+b}\times c$, or $(a+b)c$, denotes that the compound quantity $a+b$ is to be multiplied by the simple quantity c; and $\sqrt{(ab+c^2)}$, or $(ab+c^2)^{\frac{1}{2}}$, is the square root of the compound quantity $ab+c^2$.

PRACTICAL EXAMPLES.

For computing the numeral Values of Algebraic Expressions.

Supposing $a=6$, $b=5$, $c=4$ $d=1$, and $e=0$.

Then

1. $a^2+2ab-c+d=36+60-4+1=93$.
2. $2a^3-3a^2b+c^3=432-540+64=-44$.
3. $a^2\times(a+b)-2abc=36\times 11-240=156$.
4. $2a\sqrt{(b^2-ac)}+\sqrt{(2ac+c^2)}=12+1\times 8=20$.
5. $3a\sqrt{(2ac+c^2)}$, or $3a(2ac+c^2)^{\frac{1}{2}}=18\sqrt{64}=144$.
6. $\frac{2a+3c}{6d+4e}+\frac{4bc}{\sqrt{(2ac+c^2)}}=\frac{12+12}{6+0}+\frac{80}{\sqrt{(48+16)}}=\frac{24}{6}+\frac{80}{8}=14$.
7. $\sqrt{(2a^2-\sqrt{2ac+c^2})}=\sqrt{(72-\sqrt{64})}=\sqrt{(72-8)}=\sqrt{64}=8$.

EXAMPLES FOR PRACTICE.

Required the numeral values of the following expressions; supposing a, b, c, d, e, to be 6, 5, 4, 1, and 0, respectively, as above.

1. $2a^2+3bc-5d$
2. $5a^2b-10ab^2+2e$
3. $7a^2+b-c\times d+e$
4. $5\sqrt{ab}+b^2-2ab-e^2$
5. $\frac{a}{c}\times d-\frac{a-b}{d}+2a^2e$
6. $3\sqrt{c}+2a\sqrt{(2a+b-d)}$
7. $a\sqrt{a^2+b^2}+3bc\sqrt{(a^2-b^2)}$
8. $3a^2b+\sqrt[3]{(c^2+\sqrt{2ac+c^2})}$
9. $\frac{2b+c}{3a-c}-\frac{\sqrt{5b+3\sqrt{c+d}}}{2a+c}$

ADDITION.

Addition is the connecting of quantities together by means of their proper signs, and incorporating such as are like, or that can be united, into one sum; the rule for performing which is commonly divided into the three following cases: (*a*)

CASE I.

When the Quantities are like, and have like Signs.

RULE.

Add all the coefficients of the several quantities together, and to their sum annex the letter or letters belonging to each term, prefixing, when necessary, the common sign.

(*a*) The term Addition, which is generally used to denote this rule, is too scanty to express the nature of the operations that are to be performed in it; which are sometimes those of addition, and sometimes subtraction, according as the quantities are negative or positive. It should, therefore, be called by some name signifying incorporation, or striking a balance; in which case, the incongruity, here mentioned, would be removed

EXAMPLES.

1	2	3
$3a$	$-3ax$	$2b+3y$
$5a$	$-6ax$	$5b+7y$
a	$-ax$	$b+2y$
$7a$	$-2ax$	$8b+y$
$12a$	$-7ax$	$4b+4y$
$28a$	$-19ax$	$20b+17y$

4	5	6
$2ay$	$-2by^2$	$a-2x^2$
$5ay$	$-6by^2$	$a-6x^2$
$4ay$	$-by^2$	$4a-x^2$
$7ay$	$-8by^2$	$3a-5x^2$
$16ay$	$-by^2$	$7a-x^2$
$34ay$	$-18by^2$	$16a-15x^2$

7	8	9
$3ax^2$	$7x-4y$	$2a+x^2$
$2ax^2$	$x-8y$	$3a+x^2$
$12ax^2$	$3x-y$	$a+2x^2$
$9ax^2$	$x-3y$	$9a+3x^2$
$10ax^2$	$4x-y$	$4a+x^2$
$36ax^2$	$16x-17y$	$19a+8x^2$

CASE II.

When the Quantities are like, but have unlike Signs.

RULE.

Add all the affirmative coefficients into one sum, and those that are negative into another, when there are

everal of the same kind; then subtract the least of these results from the greatest, and prefix the sign of the greater to the difference, annexing the common letter or letters, as before.

EXAMPLES.

1	2	3
$-3a$	$2a-3x^2$	$6x+5ay$
$+7a$	$-7a+5x^2$	$-3x+2ay$
$+8a$	$-3a+x^2$	$x-6ay$
$-a$	$+a-3x^2$	$2x+ay$
$+11a$	$-7a$ *	$6x+2ay$

4	5	6
$-2a^2$	$2ay-7$	$-3ab+7x$
$-3a^2$	$-ay+8$	$+3ab-10x$
$-8a^2$	$+2ay-9$	$+3ab-6x$
$+10a^2$	$-3ay-11$	$-ab+2x$
$+16a^2$	$+12ay+13$	$+2ab+7x$

7	8	9
$-2a\sqrt{x}$	$-6a^2+2b$	$6ax^2+5x^{\frac{1}{2}}$
$+a\sqrt{x}$	$+2a^2-3b$	$-2ax^2-6x^{\frac{1}{2}}$
$-3a\sqrt{x}$	$-5a^2-8b$	$+3ax^2-10x^{\frac{1}{2}}$
$+7a\sqrt{x}$	$+4a^2-2b$	$-7ax^2+3x^{\frac{1}{2}}$
$-4a\sqrt{x}$	$-3a^2+9b$	$+ax^2+11x^{\frac{1}{2}}$

CASE III.

When the Quantities are unlike; or some like and others unlike.

RULE.

Collect all the like quantities together, by taking their sums or differences, as in the foregoing cases, and set down those that are unlike, one after another, with their proper signs.

EXAMPLES.

1
$$\begin{array}{r} 5xy \\ 4ax \\ -xy \\ -4ax \\ \hline 4xy \end{array}$$

2
$$\begin{array}{r} 2xy-2x^2 \\ 3x^2+xy \\ x^2+xy \\ 4x^2-3xy \\ \hline 6x^2+xy \end{array}$$

3
$$\begin{array}{r} 2ax-30 \\ 3x^2-2ax \\ 5x^2-3x^{\frac{1}{2}} \\ 3\sqrt{x}+10 \\ \hline 8x^2-20 \end{array}$$

4
$$\begin{array}{r} +ax^{\frac{1}{2}} \\ -ax^2 \\ +3ax^2 \\ -ax^{\frac{1}{2}} \\ \hline \\ \hline \end{array}$$

5
$$\begin{array}{r} 8a^2x^2-3ax \\ 7ax-5xy \\ 9xy-5ax \\ 2a^2x^2+xy \\ \hline \\ \hline \end{array}$$

6
$$\begin{array}{r} 10b^2-3a^2x \\ -b^2+2a^2x^2 \\ 50+2a^2x \\ a^2x^2+120 \\ \hline \\ \hline \end{array}$$

7
$$\begin{array}{r} +3x^2y \\ -2xy^2 \\ -3y^2x \\ -8x^2y \\ +2xy^2 \\ \hline \\ \hline \end{array}$$

8
$$\begin{array}{r} 2\sqrt{x}-18y \\ 3\sqrt{xy}+10x \\ 2x^2y+25y \\ 12xy-\sqrt{xy} \\ -8y+17x^{\frac{1}{2}} \\ \hline \\ \hline \end{array}$$

9
$$\begin{array}{r} 2a^2-3a\sqrt{x} \\ x^2-2a^{\frac{1}{2}}x^{\frac{1}{2}} \\ 3a^2-13xy \\ xy+32a^2 \\ 20-65x^2 \\ \hline \\ \hline \end{array}$$

EXAMPLES FOR PRACTICE.

1. It is required to find the sum of $\frac{a+b}{2}$ and $\frac{a-b}{2}$

2. Add $5x-3a+b+7$ and $-4a-3x+2b-9$ together.

3. Add $2a+3b-4c-9$ and $5a-3b+2c-10$ together.

4. Add $3a+2b-5$, $a+5b-c$, and $6a-2c+3$ together.

5. Add $2x^2 \times ax$, and $3x^2-bx$ together.

6. Add x^3+ax^2+bx+2 and x^3+cx^2+dx-1 together.

SUBTRACTION.

SUBTRACTION is the taking of one quantity from another; or the method of finding the difference between any two quantities of the same kind; which is performed as follows: (*b*)

RULE.

Change all the signs ($+$ and $-$) of the lower line, or quantities that are to be subtracted, into the contrary signs, or rather conceive them to be so changed, and then collect the terms together, as in the several cases of addition.

EXAMPLES.

1 $\begin{array}{r} 5a^2-2b \\ 2a^2+5b \\ \hline 3a^2-7b \end{array}$ 2 $\begin{array}{r} x^2-\ 2y+3 \\ 4x^2+\ 9y-5 \\ \hline -3x^2-11y+8 \end{array}$ 3 $\begin{array}{r} 5xy+\ 8x-2 \\ 3xy-\ 8x-7 \\ \hline 2xy+16x+5 \end{array}$

(*b*) The term subtraction, used for this rule, is liable to the same objection as that for addition; the operations to be performed, being frequently of a mixed nature, like those of the former.

4
$$\begin{array}{r} 5xy-18 \\ -xy+12 \\ \hline \\ \hline \end{array}$$

5
$$\begin{array}{r} 8y^2-2y-5 \\ -y^2+3y+2 \\ \hline \\ \hline \end{array}$$

6
$$\begin{array}{r} 10-8x-3xy \\ -x+3\ -\ xy \\ \hline \\ \hline \end{array}$$

7
$$\begin{array}{r} -5x^2y-8a \\ +3x^2y-7b \\ \hline \\ \hline \end{array}$$

8
$$\begin{array}{r} 4\sqrt{ax}-2x^2y \\ 3\sqrt{ax}-5xy^2 \\ \hline \\ \hline \end{array}$$

9
$$\begin{array}{r} 5x^2+\sqrt{x}-4y \\ 6x^2-8x-x^{\frac{1}{2}} \\ \hline \\ \hline \end{array}$$

EXAMPLES FOR PRACTICE.

1. Required the difference of $\frac{1}{2}(a+b)$ and $\frac{1}{2}(a-b)$.
2. From $3x-2a-b+7$, subtract $8-3b+a+4x$.
3. From $3a+b+c-2d$, subtract $b-8c+2d-8$.
4. From $13x^2-2ax+9b^2$, subtract $5x^2-7ax-b^2$.
5. From $20ax-5\sqrt{x}+3a$, subtract $4ax+5x^{\frac{1}{2}}-a$.
6. From $5ab+2b^2-c+bc-b$, take $b^2-2ab+bc$.
7. From $3x+ax+2$, subtract $2x^2+bx-4$.
8. From ax^3-bx^2+cx-d, subtract $bx^2+ex-2d$.

MULTIPLICATION.

Multiplication, or the method of finding the product of two or more quantities, is performed in the same manner as in arithmetic; except that it is usual, in this case, to begin the operation at the left hand, and to proceed towards the right, or contrary to the way employed in the latter.

The rule is commonly divided into three cases; in each of which, it is necessary to observe, that like signs, in multiplying, produce +, and unlike signs, —.

It is likewise to be remarked, that powers, or roots of

+ × − the product is −. but − × − the product is +.

the same quantity, are multiplied together by adding their indices: Thus,

$a \times a^2$, or $a^1 \times a^2 = a^3$; $a^2 \times a^3 = a^5$; $a^{\frac{1}{2}} \times a^{\frac{1}{3}} = a^{\frac{5}{6}}$; and $a^m \times a^n = a^{m+n}$; where m and n may be either integers or fractions.

The multiplication of compound quantities, is also, sometimes, barely denoted by writing them down, with their proper signs, under a vinculum, or a parenthesis, without performing the whole operation, as

$$3ab\,(a-b), \text{ or } 2a\sqrt{(a^2+b^2)} \text{ or } a\sqrt[3]{(a+b)}.$$

Which method is often preferable to that of executing the entire process; particularly when the product of two or more factors is to be divided by some other quantity, because, in this case, any quantity that is common to both the divisor and dividend, may be more readily suppressed; as will be evident from various instances in the following part of the work. (c)

(c) The above rule for the signs may be proved thus: If B, b, be any two quantities, of which B is the greater, and B $-\,b$ is to be multiplied by a, it is plain that the product, in this case, must be less than a B, because B $-\,b$ is less than B; and, consequently, when each of the terms of the former are multiplied by a, as above, the result will be

$$(B-b)\times b = aB - ab.$$

For if it were $aB+ab$, the product would be greater than aB, which is absurd.

Also, if B be greater than b, and A greater than a, and it is required to multiply B $-\,b$ by A $-\,a$, the result will be

$$(B-b)\times(A-a) = AB - aB - bA + ab.$$

For the product of B $-\,b$ by A is A (B $-\,b$), or AB $-$ Ab, and that of B $-\,b$ by $-a$, is $-a$(B $-\,b$), as has been before shown; whence B $-\,b$ being less than B, it is evident that $-a$ (B $-\,b$), or the part which is to be taken from A(B $-\,b$) must be less than aB, ; and consequently, since the first part of this product is $-a$B the second part must be $+ab$; for if it were $-ab$, a greater part than aB would be to be taken from A(B $-\,b$), which is absurd.

CASE I.

When the Factors are both simple Quantities.

RULE.

Multiply the coefficients of the two terms together, and to the product annex all the letters, or their powers, belonging to each, after the manner of a word; and the result, with the proper sign prefixed, will be the product required. (*d*)

EXAMPLES.

1	2	3	4
$12a$	$-2a$	$+5a$	$-9x^2$
$3b$	$+4b$	$-6x$	$-5bx$
$36ab$	$-8ab$	$-30ax$	$+45bx^3$
5	6	7	8
$7ab$	$-6a^2x$	$-2xy^2$	$-7axy$
$-5ac$	$+5x$	$-xy$	$+6ay$
$-35a^2bc$	$-30a^2x^2$	$2x^2y^3$	$42a^2xy^2$
9	10	11	12
$3a^2b$	$12a^2x$	$-6xyz$	$-a^2xy$
$2ba^2$	$-2x^2y$	$+ay^2z$	$+2xy^2$
$6a^4b^2$	$-24a^2x^3y$	$-6axy^3z^2$	$-2a^2x^2y^3$

(*d*) When any number of quantities are to be multiplied together, it is the same thing in whatever order they are placed: thus, if *ab* is to be multiplied by *c*, the product is either *abc*, *acb*, or *bca*,

CASE II.

When one of the Factors is a compound Quantity.

RULE.

Multiply every term of the compound factor, considered as a multiplicand, separately, by the multiplier, as in the former case; then these products, placed one after another, with their proper signs, will be the whole product required.

EXAMPLES.

1	2	3
$3a-2b$	$6xy-8$	a^2-2x+1
$4a$	$3x$	$4x$
$12a^2-8ab$	$18x^2y-24x$	$4a^2x-8x^2+4x$
4	5	6
$12x-ab$	$35x^2-7a$	$3y^2+y-2$
$5a$	$-2x$	xy
6 C	7 0	6
7	8	9
$13x^2-a^2b$	$25xy+3a^2$	$3x^2-xy-2y^2$
$-2a$	$13x^2$	$5x^2y$
2 6	125 225	5

&c.; although it is usual, in this case, as well as in addition and subtraction, to put them according to their rank in the alphabet. It may here also be observed, in conformity to the rule given above for the signs, that $(+a) \times (+b)$, or $(-a) \times (-b) = +ab$; and $(+a) \times (-b)$, or $(-a) \times (+b) = -ab$.

CASE III.

When both the Factors are compound Quantities.

RULE.

Multiply every term of the multiplicand separately, by each term of the multiplier, setting down the products one after another, with their proper signs; then add the several lines of products together, and their sum will be the whole product required.

EXAMPLES.

$$\begin{array}{l} a+x \\ a+x \\ \hline a^2+ax \\ \quad +ax+x^2 \\ \hline a^2+2ax+x^2 \end{array}$$

2 $$\begin{array}{l} 5x+4y \\ 3x-2y \\ \hline 15x^2+12xy \\ \quad -10xy-8y^2 \\ \hline 15x^2+2xy-8y^2 \end{array}$$

3 $$\begin{array}{l} x^2+xy-y^2 \\ x-y \\ \hline x^3+x^2y-xy^2 \\ \quad -x^2y-xy^2+y^3 \\ \hline x^3 \;\; * \;\; -2xy^2+y^3 \end{array}$$

$$\begin{array}{l} a+x \\ a-x \\ \hline a^2+ax \\ \quad -ax-x^2 \\ \hline a^2 \;\; * \;\; -x^2 \\ \hline \end{array}$$

5 $$\begin{array}{l} x^2+y \\ x^2+y \\ \hline x^4+x^2y \\ \quad +x^2y+y^2 \\ \hline x^4+2x^2y+y^2 \\ \hline \end{array}$$

6 $$\begin{array}{l} x^2+xy+y^2 \\ x-y \\ \hline x^3+x^2y+xy^2 \\ \quad -x^2y-xy^2-y^3 \\ \hline x^3 \;\; * \;\; * \;\; -y^3 \\ \hline \end{array}$$

EXAMPLES FOR PRACTICE.

1. Required the product of x^2-xy+y^2 and $x+y$.
2. Required the product of $x^3+x^2y+xy^2+y^3$ and $x-y$.

3. Required the product of x^2+xy+y^2, and x^2-xy+y^2.

4. Required the product of $3x^2-2xy+5$, and $x^2+2xy-3$.

5. Required the product of $2a^2-3ax+4x^2$, and $5a^2-6ax-2x^2$.

6. Required the product of $5x^3+4ax^2+3a^2x+a^3$, and $2x^2-3ax+a^2$.

7. Required the product of $3x^3+2x^2y^2+3y^3$, and $2x^3-3x^2y^2+5y^3$.

8. Required the product of x^3-ax^2+bx-c, and x^2-dx+e.

DIVISION.

Division is the converse of multiplication, and is performed like that of numbers; the rule being usually divided into three cases; in each of which like signs give + in the quotent and unlike signs −, as in finding their products, (e)

It is here also to be observed, that powers and roots of the same quantity, are divided by subtracting the index of the divisor from that of the dividend.

Thus, $a^3\div a^2$, or $\dfrac{a^3}{a^2}=a$; $a^{\frac{1}{2}}\div a^{\frac{1}{3}}$, or $\dfrac{a^{\frac{1}{2}}}{a^{\frac{1}{3}}}=a^{\frac{1}{6}}$;

$a^{\frac{3}{4}}\div a^{\frac{2}{3}}$ or $\dfrac{a^{\frac{3}{4}}}{a^{\frac{2}{3}}}=a^{\frac{1}{12}}$; and $a^m\div a^n$, or $\dfrac{a^m}{a^n}=a^{m-n}$.

(e) According to the rule here given for the signs, it follows that

$$\frac{+ab}{+b}=+a, \frac{-ab}{-b}=+a, \frac{-ab}{+b}=-a, \frac{+ab}{-b}=-a,$$

as will readily appear by multiplying the quotient by the divisor; the signs of the products being then the same as would take place in the former rule.

CASE I.

When the Divisor and Dividend are both simple Quantities.

RULE.

Set the dividend over the divisor, in the manner of a fraction, and reduce it to its simplest form, by cancelling the letters and figures that are common to each term.

EXAMPLES.

$6ab \div 2a$, or $\frac{6ab}{2a} = 3b$; and $12ax^2 \div 3x$, or $\frac{12ax^2}{3x} = 4ax$; $a \div a$, or $\frac{a}{a} = 1$; and $a \div -a$, or $\frac{a}{-a} = -1$.

Also $-2a \div 3a$, or $\frac{-2a}{3a} = -\frac{2}{3}$; and $9x^{\frac{1}{2}} \div 3x^{\frac{1}{4}}$, or $\frac{9x^{\frac{1}{2}}}{3x^{\frac{1}{4}}} = 3x^{\frac{1}{2} - \frac{1}{4}} = 3x^{\frac{2}{4} - \frac{1}{4}} = 3x^{\frac{1}{4}}$

1. Divide $16x^2$ by $8x$, and $12a^2x^2$ by $-8a^2x$.
2. Divide $-15ay^2$ by $3ay$, and $-18ax^2y$ by $-8ax$.
3. Divide $-\frac{2}{3}a^{\frac{1}{2}}$ by $\frac{4}{5}a^{\frac{1}{2}}$, and $ax^{\frac{1}{3}}$ by $-\frac{3}{5}a^{\frac{1}{2}}x^{\frac{1}{4}}$.

CASE II.

When the Divisor is a simple Quantity, and the Dividend a compound One.

RULE.

Divide each term of the dividend by the divisor, as in the former case; setting down such as will not divide in the simplest form they will admit of.

EXAMPLES.

$(ab+b^2)\div 2b$, or $\dfrac{ab+b^2}{2b}=\frac{1}{2}a+\frac{1}{2}b=\dfrac{a+b}{2}$

$(10ab-15ax)\div 5a$, or $\dfrac{10ab-15ax}{5a}=2b-3x.$

$(30ax-48x^2)\div 6x$, or $\dfrac{30ax-48x^2}{6x}=5a-8x$

1. Let $3x^3+6x^2+3ax-15x$ be divided by $3x$.
2. Let $3abc+12abx-9a^2b$ be divided by $3ab$.
3. Let $40a^3b^3+60a^2b^2-17\,ab$ be divided by $-ab$.
4. Let $15a^2bc-12acx^2+5ad^2$ be divided by $-5ac$.
5. Let $20ax+15ax^2+10ax+5a$ be divided by $5a$.

CASE III.

When the Divisor and Dividend are both compound Quantities.

RULE.

1. Set the quantities down in the same manner as in division of numbers, ranging the terms of each of them so, that the higher powers of one of the letters may stand before the lower.

2. Divide the first term of the dividend by the first term of the divisor, and set the result in the quotient, with its proper sign, or simply by itself, if it be affirmative.

3. Multiply the whole divisor by the term thus found; and, having subtracted the result from the dividend, bring down as many terms to the remainder as are requisite for the next operation, which perform as before; and so on, till the work is finished, as in common arithmetic.

EXAMPLES.

$$
\begin{array}{l}
x+y)x^2+2xy+y^2(x+y \\
\quad x^2+xy \\
\hline
\qquad xy+y^2 \\
\qquad xy+y^2 \\
\hline
\qquad\quad *
\end{array}
$$

$$
\begin{array}{l}
a+x)a^3+5a^2x+5ax^2+x^3(a^2+4ax+x^2 \\
\quad a^3+a^2x \\
\hline
\qquad 4a^2x+5ax^2 \\
\qquad 4a^2x+4ax^2 \\
\hline
\qquad\qquad ax^2+x^3 \\
\qquad\qquad ax^2+x^3 \\
\hline
\qquad\qquad\quad *
\end{array}
$$

$$
\begin{array}{l}
x-3)x^3-9x^2+27x-27(x^2-6x+9 \\
\quad x^3-3x^2 \\
\hline
\qquad -6x^2+27x \\
\qquad -6x^2+18x \\
\hline
\qquad\qquad 9x-27 \\
\qquad\qquad 9x-27 \\
\hline
\qquad\qquad\quad *
\end{array}
$$

$$
\begin{array}{l}
2x^2-3ax+a^2)\,4x^4-9a^2x^2+6a^3x-a^4\,(2x^2+3ax-a^2 \\
\qquad\qquad\quad 4x^4-6ax^3+2a^2x^2 \\
\qquad\qquad\qquad\qquad 6ax^3-11a^2x^2+6a^3x \\
\qquad\qquad\qquad\qquad 6ax^3-\ \ 9a^2x^2+3a^3x \\
\qquad\qquad\qquad\qquad\qquad -2a^2x^2+3a^3x-a^4 \\
\qquad\qquad\qquad\qquad\qquad -2a^2x^2+3a^3x-a^4 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad *
\end{array}
$$

NOTE 1. If the divisor be not exactly contained in the dividend, the quantity that remains after the division is finished, must be placed over the divisor at the end of the quotient, in the form of a fraction: Thus (*f*)

$$
\begin{array}{l}
a+x)\,a^3-x^3\,(a^2-ax+x^2-\dfrac{2x^3}{a+x} \\
\quad\ a^3+a^2x \\
\qquad -a^2x-x^3 \\
\qquad -a^2x-ax^2 \\
\qquad\qquad\quad ax^2-x^3 \\
\qquad\qquad\quad ax^2+x^3 \\
\qquad\qquad\qquad -2x^3
\end{array}
$$

(*f*) In the case here given, the operation of division may be considered as terminated, when the highest power of the letter, in the first, or leading term of the remainder, by which the process is regulated, is less than the power of the first term of the divisor; or when the first term of the divisor is not contained in the first term of the remainder; as the succeeding part of the quotient, after this, instead of being integral, as it ought to be, would necessarily become fractional; and, consequently, when this happens, the quotient must be completed in the manner abovementioned.

$$x+y)x^4+y^4(x^3-x^2y+xy^2-y^3+\frac{2y^4}{x+y}$$
$$\underline{x^4+x^3y}$$
$$-x^3y+y^4$$
$$\underline{-x^3y-x^2y^2}$$
$$x^2y^2+y^4$$
$$\underline{x^2y^2+xy^3}$$
$$-xy^3+y^4$$
$$\underline{-xy^3-y^4}$$
$$2y^4$$

2. The division of quantities may also be sometimes carried on, *ad infinitum*, like a decimal fraction; in which case, a few of the leading terms of the quotient will generally be sufficient to indicate the rest, without its being necessary to continue the operation: thus,

$$a+x)a\ldots.(1-\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^3}{a^3}+\frac{x^4}{a^4}\ \&c.$$
$$\underline{a+x}$$
$$-x$$
$$\underline{-x-\frac{x^2}{a}}$$
$$\frac{x^2}{a}$$
$$\underline{\frac{x^2}{a}+\frac{x^3}{a^2}}$$
$$\frac{x^3}{a^2}$$
$$\underline{\frac{x^3}{a^2}\quad\frac{x^4}{a^3}}$$

And by a process similar to the above, it may be shown that

$$\frac{a}{a-x} = 1 + \frac{x}{a} + \frac{x^2}{a^2} + \frac{x^3}{a^3} + \frac{x^4}{a^4} + \frac{x^5}{a^5} + \&c.$$

Where the law, by which either of these series may be continued at pleasure, is obvious.

EXAMPLES FOR PRACTICE.

1. Let $a^2-2ax+x^2$ be divided by $a-x$.
2. Let $x^3-3ax^2+3a^2x-a^3$ be divided by $x-\cancel{x}$. a
3. Let $a^3+5a^2x+5ax^2+x^3$ be divided by $a+x$.
4. Let $2y^3-19y^2+26y-17$ be divided by $y-8$.
5. Divide x^5+1 by $x+1$, and x^6-1 by $x-1$.
6. Divide $48x^3-76ax^2-64a^2x+105a^3$ by $2x-3a$.
7. Let $4x^4-9x^2+6x-3$ be divided by $2x^2+3x-1$.
3. Let $x^4-a^2x^2+2a^3x-a^4$ be divided by x^2-ax+a^2.
9. Let $6x^4-96$ be divided by $3x-6$, and a^5+x^5 by $a+x$.
10. Let $32x^5+243$ be divided by $2x+3$, and x^6-a^6 by $x-a$.
11. Let b^4-3y^4 be divided by $b-y$, and $a^4+4a^2b+8b^4$ by $a+2b$.
12. Let x^2+px+q be divided by $x+a$, and x^3-px^2+qx-r by $x-a$.

OF ALGEBRAIC FRACTIONS.

Algebraic fractions have the same names and rules of operation as numeral fractions in common arithmetic; and the methods of reducing them, in either of these branches, to their most convenient forms, are as follows:

Greater of Simpson med ...
9

CASE I.

To find the greatest common measure of the terms of a Fraction.

RULE.

1. Arrange the two quantities according to the order of their powers, and divide that which is of the highest dimensions by the other, having first expunged any factor, that may be contained in all the terms of the divisor, without being common to those of the dividend.

2. Divide this divisor by the remainder, simplified, if necessary, as before; and so on, for each successive remainder, and its preceding divisor, till nothing remains; when the divisor last used will be the greatest common measure required; and if such a divisor cannot be found, the terms of the fraction have no common measure.

NOTE. If any of the divisors, in the course of the operation, become negative, they may have their signs changed, or be taken affirmatively, without altering the truth of the result; and if the first term of a divisor should not be exactly contained in the first term of the dividend, the several terms of the latter may be multiplied by any number, or quantity, that will render the division complete. (*g*)

(*g*) In finding the greatest common measure of two quantities, either of them may be multiplied, or divided, by any number or quantity, which is not a divisor of the other, or that contains no factor, which is common to them both, without, in any respect, changing the result.

It may here, also, be farther added, that the common measure, or divisor, of any number of quantities, may be determined in a similar manner to that given above, by first finding the common measure of two of them, and then of that common measure and the third; and so on to the last.

EXAMPLES.

1. Required the greatest common measure of the fraction $\frac{x^4-1}{x^5+x^3}$.

$$
\begin{array}{l}
\text{Here } x^4-1)x^5+x^3(x \\
\qquad\qquad\quad x^5-x \\
\hline
\qquad\qquad\quad x^3+x \\
\qquad\quad \text{or } x^2+1 \,\big|\, x^4-1(x^2-1 \\
\qquad\qquad\qquad\quad x^4+x^2 \\
\hline
\qquad\qquad\qquad\quad -x^2-1 \\
\qquad\qquad\qquad\quad -x^2-1 \\
\hline
\qquad\qquad\qquad\qquad *
\end{array}
$$

Whence x^2+1 is the greatest common measure required.

2. Required the greatest common measure of the fraction $\frac{x^3-b^2x}{x^2+2bx+b^2}$.

$$
\begin{array}{l}
\text{Here } x^2+2bx+b^2)x^3-b^2x(x \\
\qquad\qquad\qquad\quad x^3+2bx^2+b^2x \\
\hline
\qquad\qquad\qquad\quad -2bx^2-2b^2x \\
\qquad\qquad\qquad\quad \text{or } x+b \,\big|\, x^2+2bx+b^2(x+b \\
\qquad\qquad\qquad\qquad\qquad\quad x^2+bx \\
\hline
\qquad\qquad\qquad\qquad\qquad\quad bx+b^2 \\
\qquad\qquad\qquad\qquad\qquad\quad bx+b^2 \\
\hline
\qquad\qquad\qquad\qquad\qquad\qquad *
\end{array}
$$

Where $x+b$ is the greatest common measure required.

3. Required the greatest common measure of the fraction $\dfrac{3a^2-2a-1}{4a^3-2a^2-3a+1}$.

$$3a^2-2a-1)4a^3-2a^2-3a+1$$
$$3$$
$$\overline{12a^3-6a^2-9a+3}(4a$$
$$12a^3-8a^2-4a$$
$$\overline{2a^2-5a+3})3a^2-2a-1$$
$$2$$
$$\overline{6a^2-\ \ 4a-2}(3$$
$$6a^2-15a+9$$
$$\overline{11a-11} \text{ or } a-1$$

Where, since $a-1)2a^2-5a+3(2a-3$, it follows that the last divisor $a-1$ is the common measure required.

In which case, the common process has been interrupted in the last step, merely to prevent the work overrunning the page.

4. It is required to find the greatest common measure of $\dfrac{x^3-a^3}{x^4-a^4}$. $x-a$

5. Required the greatest common measure of the fraction $\dfrac{a^4-x^4}{a^3-a^2x-ax^2+x^3}$.

6. Required the greatest common measure of the fraction $\dfrac{x^4+a^2x^2+a^4}{x^4+ax^3-a^3x-a^4}$.

7. Required the greatest common measure of the fraction $\dfrac{7a^2-23ab+6b^2}{5a^3-18a^2b+11ab^2-6b^3}$.

CASE II.

To reduce Fractions to their lowest or most simple terms.

RULE.

Divide the terms of the fraction by any number, or quantity, that will divide each of them without leaving a remainder, and the result will be the fraction required. Or find their greatest common measure, as in the last rule, by which divide both the numerator and denominator, and it will give the fraction as before.

EXAMPLES.

1. Reduce $\frac{a^2bc}{5a^2b^2}$ and $\frac{x^2}{ax+x^2}$ to their lowest terms.

Here $\frac{a^2bc}{5a^2b^2}=\frac{c}{5b}$ Ans. And $\frac{x^2}{ax+x^2}=\frac{x}{a+x}$ Ans.

2. It is required to reduce $\frac{cx+x^2}{a^2c+a^2x}$ to its lowest terms.

$$\begin{array}{r|l} \text{Here } cx+x^2 & a^2c+a^2x \\ \text{or } c+x & a^2c+a^2x(a^2 \\ & a^2c+a^2x \\ & \overline{\quad * \quad} \end{array}$$

Whence $c+x$ is the greatest common measure;

and $c+x)\ \frac{cx+x^2}{a^2c+a^2x}=\frac{x}{a^2}$ the fraction required.

3. It is required to reduce $\frac{x^3-b^2x}{x^2+2bx+b^2}$ to its lowest terms.

$$
\begin{array}{l}
x^2+2bx+b^2)x^3-b^2x(x \\
\qquad\qquad x^3+2bx^2+b^2x \\
\hline
\qquad\qquad -2bx^2-2b^2x \\
\qquad\qquad\quad \text{or } x+b \,|\, x^2+2bx+b^2(x+b \\
\qquad\qquad\qquad\qquad\quad x^2+bx \\
\hline
\qquad\qquad\qquad\qquad\qquad bx+b^2 \\
\qquad\qquad\qquad\qquad\qquad bx+b^2 \\
\hline
\qquad\qquad\qquad\qquad\qquad\quad *
\end{array}
$$

Whence $x+b$ is the greatest common measure; and

$x+b)\frac{x^3-b^2x}{x^2+2bx+b^2}=\frac{x^2-bx}{x+b}$ the fraction required.

And the same answer would have been found, if x^3-b^2x had been made the divisor instead of $x^2+2bx+b^2$.

4. It is required to reduce $\frac{x^4-a^4}{x^5-a^2x^3}$ to its lowest terms.

5. It is required to reduce $\frac{6a^2+7ax-3x^2}{6a^2+11ax+3x^2}$ to its lowest terms.

6. It is required to reduce $\frac{2x^3-16x-6}{3x^3-24x-9}$ to its lowest terms.

7. It is required to reduce $\frac{9x^5+2x^3+4x^2-x+1}{15x^4-2x^3+10x^2-x+2}$ to its lowest terms.

CASE III.

To reduce a mixed Quantity to an improper Fraction.

RULE.

Multiply the integral part by the denominator of the fraction, and to the product add the numerator, when it is affirmative, or subtract it when negative; then the result placed over the denominator, will give the improper fraction required.

EXAMPLES.

1. Reduce $3\frac{2}{5}$ and $a-\frac{b}{c}$ to improper fractions.

Here $3\frac{2}{5}=\frac{3\times 5+2}{5}=\frac{15+2}{5}=\frac{17}{5}$ Ans.

And $a-\frac{b}{c}=\frac{a\times c-b}{c}=\frac{ac-b}{c}$ Ans.

2. Reduce $x+\frac{a}{x}$ and $x-\frac{a^2-x^2}{x}$ to improper fractions.

Here $x+\frac{a}{x}=\frac{x\times x+a}{x}=\frac{x^2+a}{x}$ Ans.

And $x-\frac{a^2-x^2}{x}=\frac{x^2-a^2+x^2}{x}=\frac{2x^2-a^2}{x}$ Ans.

3. Let $1-\frac{2x}{a}$ be reduced to an improper fraction.

4. Let $5a-\frac{3x-b}{a}$ be reduced to an improper fraction.

5. Let $x-\frac{ax+x^2}{2a}$ be reduced to an improper fraction.

6. Let $5+\frac{2x-7}{3x}$ be reduced to an improper fraction.

7. Let $1-\frac{x-a-1}{a}$ be reduced to an improper fraction.

8. Let $1+2x-\frac{x-3}{5x}$ be reduced to an improper fraction.

CASE IV.

To reduce an improper fraction to a whole or mixed quantity.

RULE.

Divide the numerator by the denominator, for the integral part, and place the remainder, if any, over the denominator, for the fractional part; then the two, joined together, with the proper sign between them, will give the mixed quantity required.

EXAMPLES.

1. Reduce $\frac{27}{5}$ and $\frac{ax+a^2}{x}$ to mixed quantities.

Here $\frac{27}{5}=27\div 5=5\frac{2}{5}$ Ans.

And $\frac{ax+a^2}{x}=(ax+a^2)\div x=a+\frac{a^2}{x}$ Ans.

2. It is required to reduce the fraction $\frac{ax-x^2}{x}$ to a whole quantity.

3. It is required to reduce the fraction $\frac{ab-2a^2}{ab}$ to a mixed quantity.

4. It is required to reduce the fraction $\frac{a^2+x^2}{a-x}$ to a mixed quantity.

5. It is required to reduce the fraction $\frac{x^3-y^3}{x-y}$ to a whole quantity.

6. It is required to reduce the fraction $\frac{10x^2-5x+3}{5x}$ to a mixed quantity.

CASE V.

To reduce Fractions to other equivalent ones, that shall have a common denominator.

RULE.

Multiply each of the numerators, separately, by all the denominators, except its own, for the new numerators, and all the denominators together for a common denominator. (h)

EXAMPLES.

1. Reduce $\frac{a}{b}$ and $\frac{h}{c}$ to fractions that shall have a common denominator.

Here $\left.\begin{array}{l} a \times c = ac \\ b \times b = b^2 \end{array}\right\}$ the new numerators.

$b \times c = bc$ the common denominator.

(h) It may here be remarked, that if the numerator and denominator of a fraction be either both multiplied, or both divided, by the same number, or quantity, its value will not be altered: thus

$\frac{2}{3} = \frac{2 \times 3}{3 \times 3} = \frac{6}{9}$, and $\frac{3}{12} = \frac{3 \div 3}{12 \div 3} = \frac{1}{4}$; or $\frac{a}{b} = \frac{ac}{bc}$, and $\frac{ab}{bc} = \frac{a}{c}$

which method is often of considerable use in reducing fractions more readily to a common denominator.

Whence, $\frac{a}{b}$ and $\frac{b}{c} = \frac{ac}{bc}$ and $\frac{b^2}{bc}$, the fractions required.

2. Reduce $\frac{2x}{a}$ and $\frac{b}{c}$ to equivalent fractions having a common denominator.

3. Reduce $\frac{a}{b}$ and $\frac{a+b}{c}$ to equivalent fractions having a common denominator.

4. Reduce $\frac{3x}{2a}$, $\frac{2b}{3c}$ and d, to equivalent fractions having a common denominator.

5. Reduce $\frac{3}{4}$, $\frac{2x}{3}$ and $a + \frac{4x}{5}$, to fractions having a common denominator.

6. Reduce $\frac{a}{2}$, $\frac{3x}{7}$ and $\frac{a+x}{a-x}$, to fractions having a common denominator.

CASE VI.

To add fractional quantities together.

RULE.

Reduce the fractions, if necessary, to a common denominator; then add all the numerators together, and under their sum put the common denominator, and it will give the sum of the fractions required. (i)

(i) In the adding or subtracting of mixed quantities, it is best to bring the fractional parts only to a common denominator, and then to affix their sum or difference to the sum or difference of the integral parts, interposing the proper sign.

EXAMPLES.

1. It is required to find the sum of $\frac{x}{2}$ and $\frac{x}{3}$.

Here $\left.\begin{array}{l} x\times3=3x \\ x\times2=2x \end{array}\right\}$ the numerators.

And $2\times3=6$ the common denominator.

Whence $\frac{3x}{6}+\frac{2x}{6}=\frac{5x}{6}$, the sum required.

2. It is required to find the sum of $\frac{a}{b}$, $\frac{c}{d}$, and $\frac{e}{f}$.

Here $\left.\begin{array}{l} a\times d\times f=adf \\ c\times b\times f=cbf \\ e\times b\times d=ebd \end{array}\right\}$ the numerators.

And $b\times d\times f=bdf$ the common denominator.

Whence $\frac{adf}{bdf}+\frac{cbf}{bdf}+\frac{ebd}{bdf}=\frac{adf+cbf+ebd}{bdf}$ the sum.

3. It is required to find the sum of $a-\frac{3x^2}{b}$ and $b+\frac{2ax}{c}$. Here, taking only the fractional parts,

We shall have $\left\{\begin{array}{l} 3x^2\times c=3cx^2 \\ 2ax\times b=2abx \end{array}\right\}$ the numerators.

And $b\times c=bc$ the common denominator.

Whence $a-\frac{3cx^2}{bc}+b+\frac{2abx}{bc}=a+b+\frac{2abx-3cx^2}{bc}$ the sum.

4. It is required to find the sum of $\frac{2x}{5}$ and $\frac{5x}{7}$

5. It is required to find the sum of $\frac{3x}{2a}$ and $\frac{x}{5}$

6. It is required to find the sum of $\frac{x}{2}$, $\frac{x}{3}$ and $\frac{x}{4}$.

7. It is required to find the sum of $\frac{4x}{7}$ and $\frac{x-2}{5}$.

8. Required the sum of $2a$, $3a+\frac{2x}{5}$ and $a-\frac{8x}{9}$.

9. Required the sum of $2a+\frac{3x}{5}$, $\frac{a}{a-x}$ and $\frac{a-x}{a}$.

10. Required the sum of $5x+\frac{x+2}{3}$ and $4x-\frac{2x-3}{5x}$.

11. It is required to find the sum of $5x+\frac{2a}{3x^2}$, and $\frac{a+2x}{4x}$.

CASE VII.

To subtract one fractional Quantity from another.

RULE.

Reduce the fractions to a common denominator, if necessary, as in addition; then subtract the less numerator from the greater, and under the difference write the common denominator, and it will give the difference of the fractions required.

EXAMPLES.

1. It is required to find the difference of $\frac{2x}{3}$ and $\frac{3x}{5}$.

Here $\left.\begin{array}{l} 2x\times5=10x \\ 3x\times3=\ \ 9x \end{array}\right\}$ the numerators.

And $3\times5=15$ the common denominator.

Whence $\frac{10x}{15}-\frac{9x}{15}=\frac{x}{15}$, the difference required.

2. It is required to find the difference of $\frac{x-a}{2b}$ and $\frac{2a-4x}{3c}$

Here

$$\left.\begin{array}{l}(x-a)\times 3c=3cx-3ac\\(2a-4x)\times 2b=4ab-8bx\end{array}\right\}\text{ the numerat}^{s}.$$

And $2b\times 3c=6bc$ the common denom^{r}.

Whence $\frac{3cx-3ac}{6bc}-\frac{4ab-8bx}{6bc}=\frac{3cx-3ac-4ab+8bx}{6bc}$ the difference required.

3. Required the difference of $\frac{12x}{7}$ and $\frac{3x}{5}$

4. Required the difference of $15y$ and $\frac{1+2y}{8}$

5. Required the difference of $\frac{ax}{b-c}$ and $\frac{ax}{b+c}$

6. Required the difference of $x-\frac{x-a}{c}$ and $x+\frac{x}{2b}$

7. Required the difference of $a+\frac{a-x}{a-x}$ and $a-\frac{a+x}{a+x}$

8. It is required to find the difference of $ax+\frac{2x+7}{8}$ and $x-\frac{5x-6}{21}$.

9. It is required to find the difference of $2x+\frac{3x-5}{7}$ and $3x+\frac{11x-10}{15}$.

10. It is required to find the difference of $a+\frac{a-x}{a(a+x)}$ and $\frac{a+x}{a(a-x)}$.

CASE VIII.

To multiply fractional Quantities together.

RULE.

Multiply the numerators together for a new numerator, and the denominators for a new denominator; and the former of these, being placed over the latter, will give the product of the fractions, as required (*k*)

EXAMPLES.

1. It is required to find the product of $\frac{x}{6}$ and $\frac{2x}{9}$.

Here $\frac{x \times 2x}{6 \times 9} = \frac{2x^2}{54} = \frac{x^2}{26}$ the product required.

2. It is required to find the continued product of $\frac{x}{2}$, $\frac{4x}{5}$ and $\frac{10x}{21}$.

Here $\frac{x \times 4x \times 10x}{2 \times 5 \times 21} = \frac{40x^3}{210} = \frac{4x^3}{21}$ the product.

3. It is required to find the product of $\frac{x}{a}$ and $\frac{a+x}{a-x}$.

Here $\frac{x \times (a+x)}{a \times (a-x)} = \frac{x^2 + ax}{a^2 - ax}$ the product.

(*k*) When the numerator of one of the fractions to be multiplied, and the denominator of the other, can be divided by some quantity which is common to each of them, the quotients may be used instead of the fractions themselves.

Also, when a fraction is to be multiplied by an integer, it is the same whether the numerator be multiplied by it, or the denominator divided by it.

Or if an integer is to be multiplied by a fraction, or a fraction by an integer, the integer may be considered as having unity for its denominator, and the two be then multiplied together in the usual manner.

4. It is required to find the product of $\frac{3x}{2}$ and $\frac{5x}{3b}$

5. It is required to find the product of $\frac{2x}{5}$ and $\frac{3x^2}{2a}$

6. It is required to find the continued product of $\frac{2x}{3}$, $\frac{4x^2}{7}$ and $\frac{a}{a+x}$

7. It is required to find the continued product of $\frac{2x}{a}$ $\frac{3ab}{c}$ and $\frac{5ac}{2b}$

8. It is required to find the product of $2a+\frac{bx}{a}$ and $3a-\frac{b}{ax}$

9. It is required to find the continued product of $3x$, $\frac{x+1}{2a}$ and $\frac{x-1}{a+b}$

10. It is required to find the continued product of $\frac{a^2-x^2}{a+b}$, $\frac{a^2-b^2}{ax+x^2}$ and $a+\frac{ax}{a-x}$

CASE IX.

To divide one fractional Quantity by another.

RULE.

Multiply the denominator of the divisor by the numerator of the dividend, for the numerator; and the numerator of the divisor by the denominator of the dividend, for the denominator. Or, which is more convenient in practice multiply the dividend by the reciprocal of

the divisor, and the product will be the quotient required. (l)

EXAMPLES.

1. It is required to divide $\frac{x}{3}$ by $\frac{2x}{9}$

Here $\frac{x}{3} \div \frac{2x}{9} = \frac{x}{3} \times \frac{9}{2x} = \frac{9x}{6x} = \frac{3}{2} = 1\frac{1}{2}$ Ans.

2. It is required to divide $\frac{2a}{b}$ by $\frac{4c}{d}$

Here $\frac{2a}{b} \times \frac{d}{4c} = \frac{2ad}{4bc} = \frac{ad}{2bc}$ Ans.

3. It is required to divide $\frac{x+a}{x-b}$ by $\frac{x+b}{5x+a}$

Here $\frac{x+a}{x-b} \times \frac{5x+a}{x+b} = \frac{5x^2+6ax+a^2}{x^2-b^2}$ Ans.

4. It is required to divide $\frac{2x^2}{a^3+x^3}$ by $\frac{x}{x+a}$

Here $\frac{2x^2}{a^3+x^3} \times \frac{x+a}{x} = \frac{2x^2(x+a)}{x(a^3+x^3)} = \frac{2x}{x^2-ax+a^2}$

5. It is required to divide the fraction $\frac{7x}{5}$ by $\frac{3}{x}$

6. It is required to divide the fraction $\frac{4x^2}{7}$ by $5x$

(l) When a fraction is to be divided by an integer, it is the same whether the numerator be divided by it, or the denominator multiplied by it.

Also, when the two numerators, or the two denominators, can be divided by some common quantity, that quantity may be thrown out of each, and the quotients used instead of the fractions first proposed.

7. It is required to divide $\frac{x+1}{6}$ by $\frac{2x}{3}$.

8. It is required to divide $\frac{x}{1-x}$ by $\frac{x}{5}$.

9. It is required to divide $\frac{2ax+x^2}{c^2-x^2}$ by $\frac{x}{c-x}$.

10. It is required to divide $\frac{x^4-b^4}{x^2-2bx+b^2}$ by $\frac{x^2+bx}{x-b}$.

INVOLUTION.

INVOLUTION is the raising of powers from any proposed root; or the method of finding the square, cube, biquadrate, &c., of any given quantity.

RULE I.

Multiply the index of the quantity by the index of the power to which it is to be raised, and the result will be the power required.

Or multiply the quantity into itself as many times less one as is denoted by the index of the power, and the last product will be the answer.

Note. When the sign of the root is $+$, all the powers of it will be $+$; and when the sign is $-$, all the even powers will be $+$, and the odd powers $-$; as is evident from multiplication. (m)

(m) Any power of the product of two or more quantities is equal to the same power of each of the factors multiplied together. And any power of a fraction is equal to the same power of the numerator divided by the like power of the denominator.

Also, a^m raised to the nth power is a^{mn}; and $-a^m$ raised to the nth power is $\pm a^{mn}$, according as the index n is an even or an odd number.

EXAMPLES.

a, the root.
a^2 = square,
a^3 = cube.
a^4 = 4th power.
a^5 = 5th power.
&c.

a^2 the root.
a^4 = square.
a^6 = cube.
a^8 = 4th power.
x^{10} = 5th power.
&c.

$-3a$ the root.
$+9a^2$ = square.
$-27a^3$ = cube.
$+81a^4$ = 4th power.
&c.

$-2ax^2$ the root.
$+4a^2x^4$ = square.
$-8a^3x^6$ = cube.
$+16a^4x^8$ = 4th power.
&c.

$\frac{x}{a}$ the root.
$\frac{x^2}{a^2}$ = square.
$\frac{x^3}{a^3}$ = cube.
$\frac{x^4}{a^4}$ = 4th power.
&c.

$-\frac{2ax^2}{3b}$ the root.
$+\frac{4a^2x^4}{9b^2}$ = square.
$-\frac{8a^3x^6}{27b^3}$ = cube.
$+\frac{16a^4x^8}{81b^4}$ = 4th power.
&c.

$x-a$ the root.
$x-a$

x^2-ax
$\quad -ax+a^2$

$x^2-2ax+a^2$ square.
$x-a$

$x^3-2ax^2+a^2x$
$\quad -ax^2+2a^2x-a^3$

$x^3-3ax^2+3a^2x-a^3$ cube.

$x+a$ the root.
$x+a$

x^2+ax
$\quad +ax+a^2$

$^2+2ax+a^2$ square.
$x+a$

$x^3+2ax^2+a^2x$
$\quad +ax^2+2a^2x+a^3$

$x^3+3ax^2+3a^2x+a^3$ cube.

EXAMPLES FOR PRACTICE.

1. Required the cube, or third power, of $2a^2$.
2. Required the biquadrate, or 4th power, of $2a^2x$.
3. Required the cube, or third power, of $-\frac{2}{3}x^2y^3$
4. Required the biquadrate, or 4th power of $-\frac{3a^2x}{5b^2}$

6. Required the 4th power of $a+x$; and the 5th power of $a-y$.

RULE II.

A binomial or residual quantity, may also be raised to any power, without the trouble of continual involution, as follows:

1. Find the terms without the coefficients, by observing that the index of the first, or leading quantity, begins with that of the given power, and decreases continually by 1, in every term to the last; and that in the following quantity, the indices of the terms are 1, 2, 3, 4, &c.

2. To find the coefficients, observe that those of the first and last terms are always 1: and that the coefficient of the second term is the index of the power of the first: and for the rest, if the coefficient of any term be multiplied by the index of the leading quantity in it, and the product be divided by the number of terms to that place, it will give the coefficient of the term next following.

Note. The whole number of terms will be one more than the index of the given power; and when both terms of the root are +, all the terms of the power will be +; but if the second term be −, all the odd terms will be +, and the even terms −; or, which is the same thing, the terms will be + and − alternately. (n)

(n) The rule here laid down, which is the same in the case of integral powers as the celebrated binomial theorem of

EXAMPLES.

1. Let $a+x$ be involved, or raised to the 5th power.
Here the terms, without the coefficients, are

$$a^5,\ a^4x,\ a^3x^2,\ a^2x^3,\ ax^4,\ x^5.$$

And the coefficients, according to the rule, will be

$$1,\ 5,\ \frac{5\times4}{2},\ \frac{10\times3}{3},\ \frac{10\times2}{4},\ \frac{5\times1}{1};$$

$$\text{or } 1,\ 5,\ 10,\ 10,\ 5,\ 1.$$

Whence the entire 5th power of $a+x$ is

$$a^5+5a^4x+10a^3x^2+10a^2x^3+5ax^4+x^5$$

2. Let $a-x$ be involved, or raised, to the 6th power.
Here the terms, without their coefficients, are

$$a^6,\ a^5x,\ a^4x^2,\ a^3x^3,\ a^2x^4,\ ax^5,\ x^6.$$

And the coefficients, found as before, are

$$1,\ 6,\ \frac{6\times5}{2},\frac{15\times4}{3},\frac{20\times3}{4},\frac{15\times2}{5},\frac{6\times1}{6};$$

$$\text{or } 1,\ 6,\ 15,\ 20,\ 15,\ 6,\ 1.$$

Whence the entire 6th power of $a-x$ is

$$a^6-6a^5x+15a^4x^2-20a^3x^3+15a^2x^4-6ax^5+x^6$$

NEWTON, hereafter given, may be expressed in general terms, as follows:

$$(a+b)^m=a^m+ma^{m-1}b+\frac{m(m-1)}{2}a^{m-2}b^2+\frac{m(m-1)(m-2)}{2\ .\ 3}a^{m-3}b^3, \&c.$$

$$(a-b)m=a\ \ -ma^{m-1}b+\frac{m(m-1)}{2}a^{m-2}b^2-\frac{m(m-1)(m-2)}{2\ .\ 3}a^{m-3}b^3, \&c.$$

which formulæ will also equally hold when m is a fraction, as will be more fully explained hereafter.

It may, also, be farther observed, that the sum of the coefficients in every power, is equal to the number 2 raised to that power. Thus $1+1=2$, for the first power; $1+2+1=4=2^2$, for the square; $1+3+3+1=8=2^3$, for the cube, or third power; and so on.

3. Required the 4th power of $a+x$, and the 5th power of $a-x$.

4. Required the 6th power of $a+x$, and the 7th power of $a-y$.

5. Required the 5th power of $2+x$, and the cube of $a-bx+c$.

EVOLUTION.

EVOLUTION, or the extraction of roots, is the reverse of involution, or the raising powers; being the method of finding the square root, cube root, &c., of any given quantity.

CASE I.

To find any Root of a simple Quantity.

RULE.

Extract the root of the coefficient for the numeral part, and the root of the quantity subjoined to it for the literal part; then these, joined together, will be the root required.

And if the quantity proposed be a fraction, its root will be found, by taking the root both of its numerator and denominator.

Note. The square root, the fourth root, or any other even root, of an affirmative quantity, may be either + or −. Thus, $\sqrt{a^2}=+a$ or $-a$, and $\sqrt[4]{b^4}=+b$ or $-b$, &c. But the cube root, or any other odd root, of a quantity, will have the same sign as the quantity itself. Thus, $\sqrt[3]{a^3}=a$; $\sqrt[3]{-a^3}=-a$; and $\sqrt[5]{-a^5}=-a$, &c. (o)

(o) The reason why $+a$ and $-a$ are each the square root of a^2 is obvious, since, by the rule of multiplication, $(+a)\times(+a)$ and $(-a)\times(-a)$ are both equal to a^2.

It may here, also, be further remarked, that any even root of a negative quantity, is unassignable.

Thus, $\sqrt{-a^2}$ cannot be determined, as there is no quantity, either positive or negative, ($+$ or $-$), that, when multiplied by itself, will produce $-a^2$.

Such a quantity is called an Impossible or Imaginary quantity.

EXAMPLES.

1. Find the square root of $9x^2$; and the cube root of $8x^3$.

Here $\sqrt{9x^2}=\sqrt{9}\times\sqrt{x^2}=3\times x=3x^2$. Ans.
And $\sqrt[3]{8x^2}=\sqrt[3]{8}\times\sqrt{x^3}=2\times x=2x$. Ans.

2. It is required to find the square root of $\frac{a^2x^2}{4c^2}$, and the cube root of $-\frac{8a^3x^3}{27c^3}$.

Here $\sqrt{\frac{a^2x^2}{4c^2}}=\frac{\sqrt{a^2x^2}}{\sqrt{4c^2}}=\frac{ax}{2c}$; and $\sqrt[3]{-\frac{8a^3x^3}{27c^3}}=-\frac{2ax}{3c}$.

3. It is required to find the square root of $4a^2x^6$.
4. It is required to find the cube root of $-125a^3x^6$.
5. It is required to find the 4th root of $256a^4x^8$.
6. It is required to find the square root of $\frac{4a^4}{9x^2y^2}$.
7. It is required to find the cube root of $\frac{8a^3}{125x^6}$.
8. It is required to find the 5th root of $-\frac{32a^5x^{10}}{243}$.

And for the cube root, fifth root, &c., of a negative quantity, it is plain, from the same rule, that
$(-a)\times(-a)\times(-a)=-a^3$; and $(-a^3)\times(+a^2)=-a^5$,
And consequently $\sqrt[3]{-a^3}=-a$, and $\sqrt[5]{-a^5}=-a$.

CASE II.

To extract the square Root of a compound Quantity.

RULE.

1. Range the terms, of which the quantity is composed, according to the dimensions of some letter in them, beginning with the highest, and set the root of the first term in the quotient.

2. Subtract the square of the root, thus found, from the first term, and bring down the two next terms to the remainder, for a dividend.

3. Divide this dividend by double that part of the root already determined, and set the result both in the quotient and divisor.

4. Multiply the divisor, so increased, by the term of the root last placed in the quotient, and subtract the product from the dividend; and so on, as in common arithmetic.

EXAMPLES.

1. Extract the square root of $x^4-4x^3+6x^2-4x+1$.

$$
\begin{array}{rl}
 & x^4-4x^3+6x^2-4x+1\ (x^2-2x+1 \\
 & x^4 \\
\hline
2x^2-2x) & -4x^3+6x^2 \\
 & -4x^3+4x^2 \\
\hline
2x^2-4x+1) & 2x^2-4x+1 \\
 & 2x^2-4x+1 \\
\hline
 & *
\end{array}
$$

Ans. x^2-2x+1, the root required.

2. Extract the square root of $4a^4+12a^3x+13a^2x^2+6ax^3+x^4$.

$$
\begin{array}{l}
4a^4+12a^3x+13a^2x^2+6ax^3+x^4(2a^2+3ax+x^2 \\
4a^4 \\
4a^2+3ax)12a^3x+13a^2x^2 \\
\qquad 12a^3x+9a^2x^2 \\
4a^2+6ax+x^2)4a^2x^2+6ax^3+x^4 \\
\qquad 4a^2x^2+6ax^3+x^4 \\
\qquad *
\end{array}
$$

Note. When the quantity to be extracted has no exact root, the operation may be carried on as far as is thought necessary, or till the regularity of the terms shews the law by which the series would be continued.

EXAMPLE.

1. It is required to extract the square root of $1+x$.

$$
\begin{array}{l}
1+x\left(1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{128}\ \&c.\right. \\
1 \\
2+\frac{x}{2}\Big)\ x \\
\qquad x+\frac{x^2}{4} \\
2+x-\frac{x^2}{8}\Big)-\frac{x^2}{4} \\
\qquad -\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{64} \\
2+x-\frac{x^2}{4}+\frac{x^3}{16}\Big)\frac{x^3}{8}-\frac{x^4}{64} \\
\qquad \frac{x^3}{8}+\frac{x^4}{16}-\frac{x^5}{64}+\frac{x^6}{256} \\
\qquad -\frac{5x^4}{64}+\frac{x^5}{64}-\frac{x^6}{256}
\end{array}
$$

Here, if the numerators and denominators of the two last terms be each multiplied by 3, which will not alter their values, the root will become

$$1+\frac{x}{2}-\frac{x^2}{2.4}+\frac{3x^3}{2.4.6}-\frac{3.5x^4}{2.4.6.8}+\frac{3.5.7x^5}{2.4.6.8.10}\ \&c.$$

where the law of the series is manifest.

EXAMPLES FOR PRACTICE.

2. It is required to find the square root of $a^4+4a^3x+6a^2x^2+4ax^3+x^4$

3. It is required to find the square root of $x^4-2x^3+\frac{3}{2}x^2-\frac{1}{2}x+\frac{1}{16}$

4. It is required to find the square root of $4x^6-4x^4+12x^3+x^2-6x+9$.

5. Required the square root of $x^6+4x^5+10x^4+20x^3+25x^2+24x+16$.

6. It is required to extract the square root of a^2+b.

7. It is required to extract the square root of 2, or of $1+1$.

CASE III.

To find any Root of a compound Quantity.

RULE.

FIND the root of the first term, which place in the quotient; and having subtracted its corresponding power from that term, bring down the second term for a dividend.

Divide this by twice the part of the root above determined, for the square root; by three times the square of it, for the cube root, and so on; and the quotient will be the next term of the root.

Involve the whole of the root, thus found, to its proper power, which subtract from the given quantity, and

divide the first term of the remainder by the same divisor as before; and proceed in this manner till the whole is finished. (p)

EXAMPLES.

1. Required the square root of $a^4-2a^3x+3a^2x^2-2ax^3+x^4$.

$$a^4-2a^3x+3a^2x^2-2ax^3+x^4(a^2-ax+x^2$$
$$a^4$$
$$2a^2)-2a^3x$$
$$a^4-2a^3x+a^2x^2$$
$$2a^2)2a^2x^2$$
$$a^4-2a^3x+3a^2x^2-2ax^3+x^4$$
$$*$$

(p) When the root to be extracted consists of many terms, the following will be found a more convenient rule than that given in the text.

Arrange the expression as above, and if any of the powers of the leading quantity are wanted, supply them, and put zero for their co efficients; find the roots of the first and last terms, and set them down as the first and last terms of the whole root; then, if the root to be found is the n^{th}, raise each of these quantities to the $n-1^{th}$ power, and multiply them by n; the second term and the last but one of the given quantity, being respectively divided by the results, will give the second term and the last but one of the whole root; if this last contains any more terms, they must be put down in their proper places, with the letters a, b, c, &c., for co-efficients, and the whole expression being involved to the proper power, the quantities a, b, c, &c., will be determined by comparing the result with the given expression; but this part of the operation requires a greater proficiency in algebra than is supposed in the text.

Example.—Required the square root of

$$x^6+6x^5+13x^4+20x^3+28x^2+16x+16.$$

Here the square roots of the first and last terms are x^3 and 4, and as n is equal to 2, and $n-1$ to 1, the divisors are $2x^3$ and 8; whence $6x^5\div 2x^3=3x^2$ and $16x\div 8=2x$, are the second term, and the last but one of the root; wherefore x^3+3x^2+2x+4 is the root required.—Ed.

2. Required the cube root of $x^6+6x^5-40x^3+96x-64$.

$$
\begin{array}{l}
x^5+6x^5-40x^3+96x-64(x^2+2x-4 \\
x^6 \\
\hline
3x^4)6x^5 \\
\hline
x^5+6x^5+12x^4+8x^3 \\
\hline
\qquad 3x^4)-12x^4 \\
\hline
x^6+6x^5-40x_3+96x-64 \\
\hline
\qquad\qquad *
\end{array}
$$

Whence x^2+2x-4 is the root required.

3. Required the square root of $4a^2-12ax+9x^2$

4. Required the square root of $a^2+2ab+2ac+b^2+2bc+c^2$.

5. Required the cube root of $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.

6. Required the 4th root of $16a^4-96a^3x+216a^2x^2-216ax^3-81x^4$.

7. Required the 5th root of $32x^5-80x^4+80x^3-40x^2+10x-1$.

OF IRRATIONAL QUANTITIES, OR SURDS.

IRRATIONAL quantities, or surds, are such as have no exact root, being usually expressed by means of the radical sign, or by fractional indices; in which latter case, the numerator shows the power the quantity is to be raised to, and the denominator its root.

Thus, $\surd 2$, or $2^{\frac{1}{2}}$ is the square root of 2; and $\sqrt[3]{a^2}$ and $\surd a^3$, or $a^{\frac{2}{3}}$, and $a^{\frac{3}{2}}$, are, respectively, the squar

$a^{\frac{2}{3}} = \sqrt[3]{a^2}$ or $= (\sqrt[3]{a})^2$.

& $a^{\frac{3}{2}} = \sqrt[2]{a^3}$ or $= (\sqrt[2]{a})^3$.

the Numerator denotes a Power,
the Denominator —— a Root to be taken of it

of the cube root of a, and the cube of the square root of a, also $a^{\frac{m}{n}}$ is the mth power of the nth root of a. (q)

CASE I.

To reduce a rational Quantity to the form of a Surd.

RULE.

Raise the quantity to a power corresponding with that denoted by the index of the surd; and over this new quantity place the radical sign, or proper index, and it will be of the form required.

EXAMPLES.

1. Let 3 be reduced to the form of the square root.
   Here $3 \times 3 = 3^2 = 9$; whence $\sqrt{9}$ Ans.
2. Reduce $2x^2$ to the form of the cube root.
   Here $(2x^2)^3 = 8x^6$; whence $\sqrt[3]{8x^6}$, or $(8x^6)^{\frac{1}{3}}$ Ans.
3. Let 5 be reduced to the form of the square root.
4. Let $-3x$ be reduced to the form of the cube root.
5. Let $-2a$ be reduced to the form of the fourth root.
6. Let a^2 be reduced to the form of the fifth root, and

$\sqrt{a} + \sqrt{b}$, $\frac{\sqrt{a}}{2a}$ and $\frac{a}{b\sqrt{a}}$ to the form of the square root.

Note. Any rational quantity may be reduced by the above rule, to the form of the surd to which it is joined, and their product be then placed under the same index, or radical sign.

(q) A quantity of the kind here mentioned, as for instance $\sqrt{2}$, is called an irrational number, or a surd, because no number, either whole or fractional, can be found, which, when multiplied by itself, will produce 2. But its approximate value may be determined to any degree of exactness, by the common rule for extracting the square root, being 1 and certain non-periodic decimals, which never terminate.

EXAMPLES.

Thus $2\sqrt{2}=\sqrt{4}\times\sqrt{2}=\sqrt{(4\times 2)}=\sqrt{8}$
And $2\sqrt[3]{4}=\sqrt[3]{8}\times\sqrt[3]{4}=\sqrt[3]{(8\times 4)}=\sqrt[3]{32}$
Also $3\sqrt{a}=\sqrt{9}\times\sqrt{a}=\sqrt{(9\times a)}=\sqrt{9a}$

And $\frac{1}{2}\sqrt[3]{4a}=\sqrt[3]{\frac{1}{8}}\times\sqrt[3]{4a}=\sqrt[3]{(\frac{1}{8}\times 4a)}=\sqrt[3]{\frac{a}{2}}$

1. Let $5\sqrt{6}$ be reduced to a simple radical form.
2. Let $\frac{1}{5}\sqrt{5a}$ be reduced to a simple radical form.
3. Let $\frac{2a}{3}\sqrt[3]{\frac{9}{4a^2}}$ be reduced to a simple radical form.

CASE II.

To reduce Quantities of different Indices, to others that shall have a given Index.

RULE.

Divide the indices of the proposed quantities by the given index, and the quotients will be the new indices for those quantities.

Then, over the said quantities, with their new indices, place the given index, and they will be the equivalent quantities required.

EXAMPLES.

1. Reduce $3^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$ to quantities that shall have the index $\frac{1}{6}$.

Here $\frac{1}{2}\div\frac{1}{6}=\frac{1}{2}\times\frac{6}{1}=\frac{6}{2}=3$, the 1st index;

And $\frac{1}{3}\div\frac{1}{6}=\frac{1}{3}\times\frac{6}{1}=\frac{6}{3}=2$, the 2d index.

Whence $(3^3)^{\frac{1}{6}}$ and $(2^2)^{\frac{1}{6}}$, or $27^{\frac{1}{6}}$ and $4^{\frac{1}{6}}$, are the quantities required.

2. Reduce $5^{\frac{1}{2}}$ and $6^{\frac{1}{3}}$ to quantities that shall have the common index $\frac{1}{6}$.

3. Let $2^{\frac{1}{2}}$ and $4^{\frac{1}{4}}$ be reduced to quantities that shall have the common index $\frac{1}{8}$.

4. Let a^2 and $a^{\frac{1}{2}}$ be reduced to quantities that shall have the common index $\frac{1}{4}$.

5. Let $a^{\frac{1}{2}}$ and b be reduced to quantities that shall have the common index $\frac{1}{8}$.

Note. Surds may also be brought to a common index, by reducing the indices of the quantities to a common denominator, and then involving each of them to the power denoted by its numerator.

EXAMPLES.

1. Reduce $3^{\frac{1}{2}}$ and $4^{\frac{1}{3}}$ to quantities having a common index.

$$\text{Here } 3^{\frac{1}{2}} = 3^{\frac{3}{6}} = (3^3)^{\frac{1}{6}} = (27)^{\frac{1}{6}}$$
$$\text{And } 4^{\frac{1}{3}} = 4^{\frac{2}{6}} = (4^2)^{\frac{1}{6}} = (16)^{\frac{1}{6}}$$
$$\text{Whence } (27)^{\frac{1}{6}} \text{ and } (16)^{\frac{1}{6}} \text{ Ans.}$$

2. Reduce $4^{\frac{1}{3}}$ and $5^{\frac{1}{4}}$ to quantities that shall have a common index.

3. Reduce $a^{\frac{1}{2}}$ and $a^{\frac{1}{3}}$ to quantities that shall have a common index.

4. Reduce $a^{\frac{1}{3}}$ and $b^{\frac{1}{4}}$ to quantities that shall have a common index.

5. Reduce $a^{\frac{1}{n}}$ and $b^{\frac{1}{m}}$ to quantities that shall have a common index.

CASE III.

To reduce Surds to their most simple forms.

RULE.

Resolve the given number, or quantity, into two factors, one of which shall be the greatest power contained in it, and set the root of this power before the remaining part, with the proper radical sign between them. (*r*)

EXAMPLES.

1. Let $\sqrt{48}$ be reduced to its most simple form.
   Here $\sqrt{48} = \sqrt{(16 \times 3)} = 4\sqrt{3}$ Ans.
2. Let $\sqrt[3]{108}$ be reduced to its most simple form.
   Here $\sqrt[3]{108} = \sqrt[3]{(27 \times 4)} = 3\sqrt[3]{4}$ Ans.

Note 1. When any number, or quantity, is prefixed to the surd, that quantity must be multiplied by the root of the factor above mentioned, and the product be then joined to the other part, as before.

EXAMPLES.

1. Let $2\sqrt{32}$ be reduced to its most simple form.
   Here $2\sqrt{32} = 2\sqrt{(16 \times 2)} = 8\sqrt{2}$ Ans.
2. Let $5\sqrt[3]{24}$ be reduced to its most simple form.
   Here $5\sqrt[3]{24} = 5\sqrt[3]{(8 \times 3)} = 10\sqrt[3]{3}$ Ans.

Note 2. A fractional surd may also be reduced to a more convenient form, by multiplying both the numerator and denominator, by such a number, or quantity, as will make the denominator a complete power of the kind

---

(*r*) When the given surd involves a high number, this is most readily effected by a table of factors, from which it will immediately appear whether the surd admits of a decomposition of the kind required; if it does not admit of such a decomposition, it is already in its most simple form.

Thus, $\sqrt{15}$ cannot be reduced lower, because neither of its factors, 5, nor 3, is a square.—ED.

required; and then joining its root, with 1 put over it, as a numerator, to the other part of the surd. (*s*)

EXAMPLES.

1. Let $\sqrt{\frac{2}{7}}$ be reduced to its most simple form.

Here $\sqrt{\frac{2}{7}}=\sqrt{\frac{14}{49}}=\sqrt{\left(\frac{1}{49}\times 14\right)}=\frac{1}{7}\sqrt{14}$ Ans.

2. Let $3\sqrt[3]{\frac{2}{5}}$ be reduced to its most simple form.

Here $3\sqrt[3]{\frac{2}{5}}=3\sqrt[3]{\frac{50}{125}}=3\sqrt[3]{\left(\frac{1}{125}\times 50\right)}=\frac{3}{5}\sqrt[3]{50}$ Ans.

EXAMPLES FOR PRACTICE.

3. Let $\sqrt{125}$ be reduced to its most simple form.
4. Let $\sqrt{294}$ be reduced to its most simple form.
5. Let $\sqrt[3]{56}$ be reduced to its most simple form.
6. Let $\sqrt[3]{192}$ be reduced to its most simple form.
7. Let $7\sqrt{80}$ be reduced to its most simple form.
8. Let $9\sqrt[3]{81}$ be reduced to its most simple form.
9. Let $\frac{3}{125}\sqrt{\frac{5}{6}}$ be reduced to its most simple form.

(*s*) The utility of reducing surds to their most simple forms in order to have the answer in decimals, will be readily perceived from considering the first question above given, where it is found that $\sqrt{\frac{2}{7}}=\frac{1}{7}\sqrt{14}$; in which case it is only necessary to extract the square root of the whole number 14, (or to find it in some of the tables that have been calculated for this purpose) and then divide it by 7; whereas, we must otherwise have first divided the numerator by the denominator, and then have found the root of the quotient, for the surd part; or else have determined the root both of the numerator and denominator, and then divided the one by the other; which are each of them troublesome processes when performed by the common rules: and in the next example, for the cube root, the labour would be much greater.

10. Let $\frac{4}{7}\sqrt[3]{\frac{3}{16}}$ be reduced to its most simple form.

11. Let $\sqrt{98a^2x}$ be reduced to its most simple form.

12. Let $\sqrt{(x^3-a^2x^2)}$ be reduced to its most simple form.

CASE IV.

To add Surd Quantities together.

RULE.

When the surds are of the same kind, reduce them to their simplest forms, as in the last case; then, if the surd part be the same in them all, annex it to the sum of the rational parts, and it will give the whole sum required.

But if the quantities have different indices, or the surd part be not the same in each of them, they can only be added together by the signs $+$ and $-$

EXAMPLES.

1. It is required to find the sum of $\sqrt{27}$ and $\sqrt{48}$.

Here $\sqrt{27}=\sqrt{(9\times3)}=3\sqrt{3}$
And $\sqrt{48}=\sqrt{(16\times3)}=4\sqrt{3}$

Whence $7\sqrt{3}$ the sum.

2. It is required to find the sum of $\sqrt[3]{500}$ and $\sqrt[3]{108}$.

Here $\sqrt[3]{500}=\sqrt[3]{(125\times4)}=5\sqrt[3]{4}$
And $\sqrt[3]{108}=\sqrt[3]{(27\times4)}=3\sqrt[3]{4}$

Whence $8\sqrt[3]{4}$ the sum.

3. It is required to find the sum of $4\sqrt{147}$ and $3\sqrt{75}$.

Here $4\sqrt{147}=4\sqrt{(49\times3)}=28\sqrt{3}$
And $3\sqrt{75}=3\sqrt{(25\times3)}=15\sqrt{3}$

Whence $43\sqrt{3}$ the sum.

4. It is required to find the sum of $3\sqrt{\frac{2}{5}}$ and $2\sqrt{\frac{1}{10}}$

Here $3\sqrt{\frac{2}{5}} = 3\sqrt{\frac{10}{25}} = \frac{3}{5}\sqrt{10}$

And $2\sqrt{\frac{1}{10}} = 2\sqrt{\frac{10}{100}} = \frac{2}{10}\sqrt{10}$

Whence $\frac{4}{5}\sqrt{10}$ the sum.

EXAMPLES FOR PRACTICE.

5. It is required to find the sum of $\sqrt{72}$ and $\sqrt{128}$.
6. It is required to find the sum of $\sqrt{180}$ and $\sqrt{405}$.
7. It is required to find the sum of $3\sqrt[3]{40}$ and $\sqrt[3]{135}$.
8. It is required to find the sum of $4\sqrt[3]{54}$ and $5\sqrt[3]{128}$.
9. It is required to find the sum of $9\sqrt{243}$ and $10\sqrt{363}$.
10. It is required to find the sum of $3\sqrt{\frac{2}{3}}$ and $7\sqrt{\frac{27}{50}}$
11. It is required to find the sum of $12\sqrt[3]{\frac{1}{4}}$ and $3\sqrt[3]{\frac{1}{32}}$
12. It is required to find the sum of $\frac{1}{2}\sqrt{a^2b}$ and $\frac{1}{3}\sqrt{4bx^4}$.

CASE V.

To find the Difference of Surd Quantities.

RULE.

When the surds are of the same kind, prepare the quantities as in the last rule; then the difference of the rational parts annexed to the common surd, will give the whole difference required.

But if the quantities have different indices, or the surd part be not the same in each of them; they can only be subtracted by means of the sign $-$.

1. It is required to find the difference of $\sqrt{448}$ and $\sqrt{112}$.

$$\begin{aligned} \text{Here } \sqrt{448} &= \sqrt{(64\times 7)} = 8\sqrt{7} \\ \text{And } \sqrt{112} &= \sqrt{(16\times 7)} = 4\sqrt{7} \end{aligned}$$

Whence $4\sqrt{7}$ the difference.

2. It is required to find the difference of $\sqrt[3]{192}$ and $\sqrt[3]{24}$.

$$\begin{aligned} \text{Here } \sqrt[3]{192} &= \sqrt[3]{(64\times 3)} = 4\sqrt[3]{3} \\ \text{And } \sqrt[3]{24} &= \sqrt[3]{(8\times 3)} = 2\sqrt[3]{3} \end{aligned}$$

Whence $2\sqrt[3]{3}$ the difference.

3. It is required to find the difference of $5\sqrt{20}$ and $3\sqrt{45}$.

$$\begin{aligned} \text{Here } 5\sqrt{20} &= 5\sqrt{(4\times 5)} = 10\sqrt{5} \\ \text{And } 3\sqrt{45} &= 3\sqrt{(9\times 5)} = 9\sqrt{5} \end{aligned}$$

Whence $\sqrt{5}$ the difference.

4. It is required to find the difference of $\frac{3}{4}\sqrt{\frac{2}{3}}$, and $\frac{2}{5}\sqrt{\frac{1}{6}}$

$$\begin{aligned} \text{Here } \frac{3}{4}\sqrt{\frac{2}{3}} &= \frac{3}{4}\sqrt{\frac{6}{9}} = \frac{3}{12}\sqrt{6} = \frac{1}{4}\sqrt{6} \\ \text{And } \frac{2}{5}\sqrt{\frac{1}{6}} &= \frac{2}{5}\sqrt{\frac{6}{36}} = \frac{2}{30}\sqrt{6} = \frac{1}{15}\sqrt{6} \end{aligned}$$

Whence $\frac{11}{60}\sqrt{6}$ the difference.

EXAMPLES FOR PRACTICE.

1. It is required to find the difference of $2\sqrt{50}$ and $\sqrt{18}$.
2. It is required to find the difference of $\sqrt[3]{320}$ and $\sqrt[3]{40}$.
3. It is required to find the difference of $\sqrt{\frac{3}{5}}$ and $\sqrt{\frac{5}{27}}$.
4. It is required to find the difference of $2\sqrt{\frac{1}{2}}$ and $\sqrt{8}$.
5. It is required to find the difference of $3\sqrt[3]{\frac{1}{3}}$ and $\sqrt[3]{72}$.
6. It is required to find the difference of $\sqrt[3]{\frac{2}{3}}$ and $\sqrt[3]{\frac{9}{32}}$.
7. It is required to find the difference of $\sqrt{80a^4x}$ and $\sqrt{20a^2x^3}$.
8. It is required to find the difference of $8\sqrt[3]{a^3b}$ and $2\sqrt[3]{a^6b}$.

CASE VI.

To multiply Surd Quantities together.

RULE.

When the surds are of the same kind, find the product of the rational parts, and the product of the surds, and the two joined together, with their common radical sign between them, will give the whole product required; which may be reduced to its most simple form by Case III.

But if the surds are of different kinds, they must be reduced to a common index, and then multiplied together as usual.

It is also to be observed, as before mentioned, that the product of different powers, or roots, of the same quantity, is found by adding their indices.

EXAMPLES.

1. It is required to find the product of $3\sqrt{8}$ and $2\sqrt{6}$.

Here $3\sqrt{8}$

Multiplied $2\sqrt{6}$

Gives $6\sqrt{48}=6\sqrt{(16\times 3)}=24\sqrt{3}$ Ans.

2. It is required to find the product of $\frac{1}{2}\sqrt[3]{\frac{2}{3}}$ and $\frac{3}{4}\sqrt[3]{\frac{5}{6}}$

Here $\frac{1}{2}\sqrt[3]{\frac{2}{3}}$

Multiplied $\frac{3}{4}\sqrt[3]{\frac{5}{6}}$

Gives $\frac{3}{8}\sqrt[3]{\frac{10}{18}}=\frac{3}{8}\sqrt[3]{\frac{5}{9}}=\frac{3}{8}\sqrt[3]{\frac{15}{27}}=\frac{1}{8}\sqrt[3]{15}$

3. It is required to find the product of $2^{\frac{1}{2}}$ and $3^{\frac{1}{3}}$.

Here $2^{\frac{1}{2}}=2^{\frac{3}{6}}=(2^3)^{\frac{1}{6}}=8^{\frac{1}{6}}$

And $3^{\frac{1}{3}}=3^{\frac{2}{6}}=(3^2)^{\frac{1}{6}}=9^{\frac{1}{6}}$

Whence $(72)^{\frac{1}{6}}$ Ans.

4. It is required to find the product of $5\sqrt{a}$ and $3\sqrt[3]{a}$.

Here $5\sqrt{a}=5a^{\frac{1}{2}}=5a^{\frac{3}{6}}$

And $3\sqrt[3]{a}=3a^{\frac{1}{3}}=3a^{\frac{2}{6}}$

Whence $15a^{\frac{5}{6}}=15(a^5)^{\frac{1}{6}}$ or $15\sqrt[6]{a^5}$ Ans.

EXAMPLES FOR PRACTICE.

5. It is required to find the product of $5\sqrt{8}$ and $3\sqrt{5}$.
6. It is required to find the product of $\sqrt[3]{18}$ and $5\sqrt[3]{4}$.

7. Required the product of $\frac{1}{4}\sqrt[3]{6}$ and $\frac{2}{15}\sqrt[3]{9}$.

8. Required the product of $\frac{1}{2}\sqrt[3]{18}$ and $5\sqrt[3]{20}$.

9. Required the product of $2\sqrt{3}$ and $13\frac{1}{2}\sqrt[3]{5}$.
10. Required the product of $72\frac{1}{4}\sqrt[3]{a^2}$ and $120\frac{1}{2}\sqrt[4]{a}$.
11. Required the product of $4+2\sqrt{2}$ and $2-\sqrt{2}$.
12. Required the product of $(a+b)^{\frac{1}{n}}$ and $(a+b)^{\frac{1}{m}}$

CASE VII.

To divide one Surd Quantity by another.

RULE.

When the surds are of the same kind, find the quotient of the rational parts, and the quotient of the surds, and the two joined together, with their common radical sign between them, will give the whole quotient required.

But if the surds are of different kinds, they must be reduced to a common index, and then be divided as before.

It is also to be observed, that the quotient of different powers or roots of the same quantity, is found by subtracting their indices.

EXAMPLES.

1. It is required to divide $8\sqrt{108}$ by $2\sqrt{6}$.

Here $\frac{8\sqrt{108}}{2\sqrt{6}}=4\sqrt{18}=4\sqrt{(9\times2)}=12\sqrt{2}$ Ans.

2. It is required to divide $8\sqrt[3]{512}$ by $4\sqrt[3]{2}$.

Here $\frac{8\sqrt[3]{512}}{4\sqrt[3]{2}}=2\sqrt[3]{256}=2\sqrt[3]{(64\times4)}=8\sqrt[3]{4}$ Ans.

3. It is required to divide $\frac{1}{2}\sqrt{5}$ by $\frac{1}{3}\sqrt{2}$

Here $\frac{\frac{1}{2}\sqrt{5}}{\frac{1}{3}\sqrt{2}}=\frac{3}{2}\sqrt{\frac{5}{2}}=\frac{3}{2}\sqrt{\frac{10}{4}}=\frac{3}{4}\sqrt{10}$ Ans.

4. It is required to divide $\sqrt{7}$ by $\sqrt[3]{7}$.

Here $\frac{\sqrt{7}}{\sqrt[3]{7}}=\frac{7^{\frac{1}{2}}}{7^{\frac{1}{3}}}=\frac{7^{\frac{3}{6}}}{7^{\frac{2}{6}}}=7^{\frac{3}{6}-\frac{2}{6}}=7^{\frac{1}{6}}$ Ans.

5. It is required to divide $6\sqrt{54}$ by $3\sqrt{2}$.

6. It is required to divide $4\sqrt[3]{72}$ by $2\sqrt[3]{18}$.

7. It is required to divide $5\frac{3}{4}\sqrt{\frac{1}{135}}$ by $\frac{2}{3}\sqrt{\frac{1}{5}}$

8. It is required to divide $3\frac{5}{7}\sqrt[3]{\frac{2}{3}}$ by $2\frac{2}{5}\sqrt[3]{\frac{3}{4}}$

9. It is required to divide $4\frac{1}{2}\sqrt{a}$ by $2\frac{2}{3}\sqrt[3]{ab}$.

10. It is required to divide $32\frac{2}{5}\sqrt{a}$ by $13\frac{3}{4}\sqrt[3]{a}$

11. It is required to divide $9\frac{3}{8}a^{\frac{1}{n}}$ by $4\frac{9}{11}a^{\frac{1}{m}}$

12. Let $\sqrt{20}+\sqrt{12}$ be divided by $\sqrt{5}-\sqrt{3}$.

Note. Since the division of surds is performed by subtracting their indices, it is evident that the denominator of any fraction may be taken into the numerator, or the numerator into the denominator, by changing the sign of its index.

Also, since $\frac{a^m}{a^m}=1$, or $=a^{m-m}=a^0$, it follows that the expression a^0 is a symbol equivalent to unity, and consequently, that it may be always replaced by 1 whenever it occurs. (t)

(t) To what is above said, we may also farther observe,

1. That 0 added to or subtracted from any quantity, makes it neither greater or less; that is,

EXAMPLES.

1. Thus, $\frac{1}{a}=\frac{a^{-1}}{1}$, or a^{-1}, and $\frac{1}{a^n}=\frac{a^{-n}}{1}$, a^{-n}

2. Also, $\frac{b}{a^2}=\frac{ba^{-2}}{1}$, or ba^{-2}; and $\frac{a^{-n}}{b^{-m}}=\frac{b^m}{a^n}$

3. Let $\frac{1}{a^2}$ be expressed with a negative index.

4. Let $a^{-\frac{1}{2}}$ be expressed with a positive index.

5. Let $\frac{1}{a+x}$ be expressed with a negative index.

6. Let $a(a^2-x^2)^{-\frac{1}{3}}$ be expressed with a positive index.

$$a+0=a, \text{ and } a-0=a.$$

2. Also, if nought be multiplied or divided by any quantity, both the product and quotient will be nought; because any number of times 0, or any part of 0, is 0; that is,

$$0\times a, \text{ or } a\times 0=0, \text{ and } \frac{0}{a}=0.$$

3. From this it likewise follows, that nought divided by nought, may be a finite quantity.

For since $0\times a=0$, or $0=0\times a$, it is evident, that $\frac{0}{0}=a$.

4. Farther, if any finite quantity be divided by 0, the quotient will be infinite; and if it be divided by an infinite quantity, the quotient will be 0.

For let $\frac{a}{b}=q$; then, if a be supposed to remain constant, it is plain, the less b is, the greater will be the quotient q; whence, if b be indefinitely small, q will be indefinitely great; and, consequently, when b is 0, the quotient q will be infinite: that is,

$$\frac{a}{0}=\infty, \text{ and } \frac{a}{\infty}=0.$$

Which properties are of frequent occurrence in some of the higher parts of the science, and should be carefully remembered.

CASE VIII.

To involve or raise surd quantities to any power.

RULE.

When the surd is a simple quantity, multiply its index by 2 for the square, by 3 for the cube, *&c.*, and it will give the power of the surd part, which being annexed to the proper power of the rational part, will give the whole power required. And if it be a compound quantity, multiply it by itself the proper number of times, according to the usual rule. (u)

EXAMPLES.

1. It is required to find the square of $\frac{2}{3}a^{\frac{1}{3}}$

Here $\left(\frac{2}{3}a^{\frac{1}{3}}\right)^2 = \frac{4}{9}a^{\frac{1}{3}\times 2} = \frac{4}{9}a^{\frac{2}{3}} = \frac{4}{9}\sqrt[3]{a^2}$ Ans.

2. It is required to find the cube of $\frac{2}{3}\sqrt{3}$.

Here $\frac{8}{27}\times 3^{\frac{3}{2}} = \frac{8}{27}\sqrt{27} = \frac{8}{27}\sqrt{(9\times 3)} = \frac{8}{9}\sqrt{3}$ Ans.

3. It is required to find the square of $3\sqrt[3]{3}$.
4. It is required to find the cube of $17\sqrt{21}$.
5. It is required to find the 4th power of $\frac{1}{6}\sqrt{6}$.
6. It is required to find the square of $3+2\sqrt{5}$.
7. It is required to find the cube of $\sqrt{x}+3\sqrt{y}$.
8. It is required to find the 4th power of $\sqrt{3}-\sqrt{2}$.

(u) When any quantity that is affected with the sign of the square root, is to be raised to the second power, or squared, it is done by suppressing the sign. Thus, $(\sqrt{a})^2$, or $\sqrt{a}\times\sqrt{a}=a$; and $(\sqrt{a+b})^2$, or $\sqrt{(a+b)}\times\sqrt{(a+b)}=a+b$.

CASE IX.

To find the Roots of Surd Quantities.

RULE.

When the surd is a simple quantity, multiply its index by $\frac{1}{2}$ for the square root, by $\frac{1}{3}$ for the cube root, *&c.*, and it will give the root of the surd part; which being annexed to the root of the rational part, will give the whole root required. And if it be a compound quantity, find its root by the usual rule. (x)

EXAMPLES.

1. It is required to find the square root of $9\sqrt[3]{3}$.

Here $(9\sqrt[3]{3})^{\frac{1}{2}} = 9^{\frac{1}{2}} \times 3^{\frac{1}{3}\times\frac{1}{2}} = 9^{\frac{1}{2}} \times 3^{\frac{1}{6}} = 3\sqrt[6]{3}$ Ans.

2. It is required to find the cube root of $\frac{1}{8}\sqrt{2}$.

Here $\left(\frac{1}{8}\sqrt{2}\right)^{\frac{1}{3}} = \left(\frac{1}{8}\right)^{\frac{1}{3}} \times (2^{\frac{1}{2}\times\frac{1}{3}}) = \frac{1}{2}(2^{\frac{1}{6}}) = \frac{1}{2}\sqrt[6]{2}$ Ans.

3. It is required to find the square root of 10^3.

4. It is required to find the cube root of $\frac{8}{27}a^4$.

5. It is required to find the 4th root of $\frac{16}{81}a^{\frac{2}{3}}$

(x) The nth root of the mth power of any number a, or the mth power of the nth root of a, is $a^{\frac{m}{n}}$.

Also, the nth root of the mth root of any number a, or the mth root of the nth root of a, is $a^{\frac{1}{mn}}$.

From which last expression, it appears that the square root of the square root of a is the 4th root of a; and that the cube root of the square root of a, or the square root of the cube root of a, is the 6th root of a; and so on for the fourth, fifth, or any other numerical root of this kind.

6. It is required to find the cube root of $\frac{a}{3}\sqrt{\frac{a}{3}}$

7. It is required to find the square root of $x^2-4x\sqrt{a}+4a$.

8. It is required to find the square root of $a+2\sqrt{ab}+b$.

CASE X.

To transform a binomial, or a residual Surd, into a general Surd.

RULE.

Involve the given binomial, or residual, to a power corresponding with that denoted by the surd; then set the radical sign of the same root over it, and it will be the general surd required.

EXAMPLES.

1. It is required to reduce $2+\sqrt{3}$ to a general surd.

Here $(2+\sqrt{3})^2=4+3+4\sqrt{3}=7+4\sqrt{3}$; therefore $2+\sqrt{3}=\sqrt{(7+4\sqrt{3})}$, the answer.

2. It is required to reduce $\sqrt{2}+\sqrt{3}$ to a general surd.

Here $(\sqrt{2}+\sqrt{3})^2=2+3+2\sqrt{6}=5+2\sqrt{6}$; therefore $\sqrt{2}+\sqrt{3}=\sqrt{(5+2\sqrt{6})}$, the answer.

3. It is required to reduce $\sqrt[3]{2}+\sqrt[3]{4}$ to a general surd.

Here $(\sqrt[3]{2}+\sqrt[3]{4})^3=6+6\sqrt[3]{2}+6\sqrt[3]{4}$; therefore $\sqrt[3]{2}+\sqrt[3]{4}=\sqrt[3]{\{6(1+\sqrt{2}+\sqrt{4})\}}$, the answer.

4. It is required to reduce $3-\sqrt{5}$ to a general surd.

5. It is required to reduce $\sqrt{2}-2\sqrt{6}$ to a general surd.

6. It is required to reduce $4-\sqrt{7}$ to a general surd.

7. It is required to reduce $2\sqrt[3]{3}-\sqrt[3]{9}$ to a general surd.

CASE XI.

To extract the square Root of a binomial, or residual Surd.

RULE.

Substitute the numbers, or parts, of which the given surd is composed, in the place of the letters, in one of the two following formulæ, according as it is a binomial or a residual, and it will give the root required.

$$\sqrt{(a+\sqrt{b})}=\sqrt{\{\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{(a^2-b)}\}}+\sqrt{\{\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{(a^2-b)}\}}$$

$$\sqrt{(a-\sqrt{b})}=\sqrt{\{\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{(a^2-b)}\}}-\sqrt{\{\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{(a^2-b)}\}}$$

Where it is to be observed, that if both a and $\sqrt{(a^2-b)}$, in these formulæ, be rational quantities, the root will consist either of two surds, or of a rational part and a surd, which are the only cases of the rule that are useful.

EXAMPLES.

1. It is required to find the square root of $11+\sqrt{72}$, or $\sqrt{(11+6\sqrt{2})}$.

Here, $\sqrt{\{\frac{1}{2}a+\frac{1}{2}\sqrt{(a^2-b)}\}}=\sqrt{\{\frac{11}{2}+\frac{1}{2}\sqrt{(121-72)}\}}=\sqrt{(\frac{11}{2}+\frac{7}{2})}=3$;

And, $\sqrt{\{\frac{1}{2}a-\frac{1}{2}\sqrt{(a^2-b)}\}}=\sqrt{\{\frac{11}{2}-\frac{1}{2}\sqrt{(121-72)}\}}=\sqrt{(\frac{11}{2}-\frac{7}{2})}=\sqrt{2}$

Whence $\sqrt{(11+6\sqrt{2})}=3+\sqrt{2}$, the answer required.

2. It is required to find the square root of $3-2\sqrt{2}$.

Here, $\sqrt{\{\frac{1}{2}a+\frac{1}{2}\sqrt{(a^2-b)}\}}=\sqrt{\{\frac{3}{2}+\frac{1}{2}\sqrt{(9-8)}}=\sqrt{(\frac{3}{2}+\frac{1}{2})}=\sqrt{2}$; and $-\sqrt{\{\frac{1}{2}a-\frac{1}{2}\sqrt{(a^2-b)}\}}=-\sqrt{\{\frac{3}{2}-\frac{1}{2}\sqrt{(9-8)}\}}=-\sqrt{(\frac{3}{2}-\frac{1}{2})}=-1$;

Whence $3-2\sqrt{2}=\sqrt{2}-1$, the answer required.

3. It is required to find the square root of $6 \pm 2\sqrt{5}$.
Ans. $\sqrt{5}+1$, or $\sqrt{5}-1$.

4. It is required to find the square root of $23 \pm 8\sqrt{7}$.
Ans. $4+\sqrt{7}$, or $4-\sqrt{7}$.

5. It is required to find the square root of $36 \pm 10\sqrt{11}$.

6. It is required to find the square root of $33 \pm 12\sqrt{6}$.

CASE XII.

To find such a Multiplier, or Multipliers, as will make any binomial Surd rational.

RULE.

1. When one or both of the terms are any even roots, multiply the given binomial, or residual, by the same expression, with the sign of one of its terms changed; and repeat the operation in the same way, as long as there are surds, when the last result will be rational.

2. When the terms of the binomial surd are odd roots, the rule becomes more complicated; but for the sum or difference of two cube roots, which is one of the most useful cases, the multiplier will be a trinomial surd, consisting of the squares of the two given terms and their product, with its sign changed *.

EXAMPLES.

1. To find a multiplier that shall render $5+\sqrt{3}$ rational.

Given surd $5+\sqrt{3}$
Multiplier $5-\sqrt{3}$

Product $25-3=22$, as required.

* The following rule will be found more convenient than that given in the text, and answers equally for odd or even roots.

Rule.—Expand the given surd, with the sign of one of its terms changed, to a power one degree less than the denominator of the surd, the result, neglecting the co-efficients, is the multiplier required.—Ed.

2. To find a multiplier that shall make $\sqrt{5}+\sqrt{3}$ rational.

$$\begin{array}{ll} \text{Given surd} & \sqrt{5}+\sqrt{3} \\ \text{Multiplier} & \sqrt{5}-\sqrt{3} \\ \hline \text{Product} & 5-3=2, \text{ as required.} \end{array}$$

3. To find multipliers that shall make $\sqrt[4]{5}+\sqrt[4]{3}$ rational.

$$\begin{array}{ll} \text{Given surd} & \sqrt[4]{5}+\sqrt[4]{3} \\ \text{1st multiplier} & \sqrt[4]{5}-\sqrt[4]{3} \\ \hline \text{1st product} & \sqrt{5}-\sqrt{3} \\ \text{2nd multiplier} & \sqrt{5}+\sqrt{3} \\ \hline \text{2nd product} & 5-3=2, \text{ as required.} \end{array}$$

4. To find a multiplier that shall make $\sqrt[3]{7}+\sqrt[3]{3}$ rational.

$$\begin{array}{ll} \text{Given surd} & \sqrt[3]{7}+\sqrt[3]{3} \\ \text{Multiplier} & \sqrt[3]{7^2}-\sqrt[3]{(7\times3)}+\sqrt[3]{3^2} \\ \hline & 7+\sqrt[3]{(3\times7^2)} \\ & \quad -\sqrt[3]{(3\times7^2)}-\sqrt[3]{(7\times3^2)} \\ & \qquad +\sqrt[3]{(7\times3^2)}+3 \\ \hline \text{Product} & 7+3=10, \text{ as was required.} \end{array}$$

5. To find a multiplier that shall make $\sqrt{5}-\sqrt{x}$ rational.

6. To find a multiplier that shall make $\sqrt{a}+\sqrt{b}$ rational.

7. To find multipliers that shall make $a+\sqrt{b}$ rational.

8. It is required to find a multiplier that shall make $1-\sqrt[3]{2a}$ rational.

9. It is required to find a multiplier that shall make $\sqrt[3]{3}-\frac{1}{2}\sqrt[3]{2}$ rational.

CASE XIII.

To reduce a Fraction, whose Denominator is either a ngle + *simple or a compound Surd, to another that shall have a rational Denominator.*

RULE.

1. When any simple fraction is of the form $\frac{b}{\sqrt{a}}$, multiply each of its terms by $\sqrt{a}$, and the resulting fraction will be $\frac{b\sqrt{a}}{a}$. $= \frac{b}{a}\sqrt{a}$.

Or when it is of the form $\frac{b}{\sqrt[3]{a}}$, multiply them by $\sqrt[3]{a^2}$, and the result will be $\frac{b\sqrt[3]{a^2}}{a}$. $= \frac{b}{a}\sqrt[3]{a^2}$.

And for the general form $\frac{b}{\sqrt[n]{a^m}}$, multiply by $\sqrt[n]{a^{n-r}}$, and the result will be $\frac{b\sqrt[n]{a^{n-m}}}{a}$.

2. If it be a compound surd, find such a multiplier by the last rule, as will make the denominator rational; and multiply both the numerator and denominator by it, and the result will be the fraction required.

EXAMPLES.

1. Reduce the fractions $\frac{2}{\sqrt{3}}$ and $\frac{3}{\frac{1}{2}\sqrt[4]{5}}$, to others that shall have rational denominators.

Here $\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$; and $\frac{3}{\frac{1}{2}\sqrt[4]{5}} = \frac{3}{\frac{1}{2}\sqrt[4]{5}} \times \frac{\sqrt[4]{5^3}}{\sqrt[4]{5^3}} = \frac{3\sqrt[4]{5^3}}{\frac{1}{2}\times 5} = \frac{6\sqrt[4]{5^3}}{5} = \frac{6}{5}\sqrt[4]{125}$, the answer required.

2. Reduce $\frac{3}{\sqrt{5}-\sqrt{2}}$ to a fraction, whose denominator shall be rational.

Here

$$\frac{3}{\sqrt{5}-\sqrt{2}}\times\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}+\sqrt{2}}=\frac{3\sqrt{5}+3\sqrt{2}}{5-2}=\frac{3\sqrt{5}+3\sqrt{2}}{3}=$$

$\frac{\sqrt{5}+\sqrt{2}}{1}=\sqrt{5}+\sqrt{2}$, the answer required.

3. Reduce $\frac{\sqrt{2}}{3-\sqrt{2}}$ to a fraction, whose denominator shall be rational.

Here

$$\frac{\sqrt{2}}{3-\sqrt{2}}=\frac{\sqrt{2}\times(3+\sqrt{2})}{(3-\sqrt{2})\times(3+\sqrt{2})}=\frac{3\sqrt{2}+2}{9-2}=\frac{2+3\sqrt{2}}{7}=$$

$\frac{2}{7}+\frac{3}{7}\sqrt{2}$ the answer required.

4. Reduce $\frac{\sqrt{6}}{\sqrt{7}+\sqrt{3}}$ to a fraction, that shall have a rational denominator.

5. Reduce $\frac{x}{3+\sqrt{x}}$ to a fraction, that shall have a rational denominator.

6. Reduce $\frac{a-\sqrt{b}}{a+\sqrt{b}}$ to a fraction, the denominator of which shall be rational.

7. Reduce $\frac{10}{\sqrt[3]{7}-\sqrt[3]{5}}$ to a fraction, that shall have a rational denominator.

8. Reduce $\frac{\sqrt[3]{3}}{\sqrt[3]{9}+\sqrt[3]{10}}$ to a fraction, that shall have a rational denominator.

9. Reduce $\frac{4}{\sqrt[4]{4}+\sqrt[4]{5}}$ to a fraction, that shall have a rational denominator.

OF
ARITHMETICAL PROPORTION AND PROGRESSION.

Arithmetical Proportion, is the relation which two numbers, or quantities, of the same kind, have to two others, when the difference of the first pair is equal to that of the second.

Hence, three quantities are in arithmetical proportion, when the difference of the first and second is equal to the difference of the second and third.

Thus, 2, 4, 6, and a, $a+b$, $a+2b$, are quantities in arithmetical proportion.

And four quantities are in arithmetical proportion, when the difference of the first and second is equal to the difference of the third and fourth.

Thus, 3, 7, 12, 16, and a, $a+b$, c, $c+b$, are quantities in arithmetical proportion.

Arithmetical Progression is when a series of numbers, or quantities, increase or decrease by the same common difference.

Thus, 1, 3, 5, 7, 9, *&c.*, and a, $a+d$, $a+2d$, $a+3d$, *&c.*, are increasing series in arithmetical progression, the common differences of which are 2 and d.

And 15, 12, 9, 6, *&c.*, and a, $a-d$, $a-2d$, $a-3d$, *&c.* are decreasing series in arithmetical progression, the common differences of which are 3 and d.

The most useful properties of arithmetical proportion and progression are contained in the following theorems:

1. If four quantities are in arithmetical proportion, the sum of the two extremes will be equal to the sum of the two means.

Thus, if the proportionals be 2, 5, 7, 10, or a, b, c, d; then will $2+10=5+7$, and $a+d=b+c$.

2. And if three quantities be in arithmetical propor-

tion, the sum of the two extremes will be double the mean.

Thus, if the proportionals be 3, 6, 9, or a, b, c, then will $3+9=2\times6=12$, and $a+c=2b$.

3. Hence an arithmetical mean between any two quantities is equal to half the sum of those quantities.

Thus, an arithmetical mean between 2 and 4 is $=\frac{2+4}{2}=3$; and between 5 and 6 it is $=\frac{5+6}{2}=5\frac{1}{2}$.

And an arithmetical mean between a and b is $\frac{a+b}{2}$.

4. In any continued arithmetical progression, the sum of the two extremes is equal to the sum of any two terms that are equally distant from them, or to double the middle term, when the number of terms is odd.

Thus, if the series be 2, 4, 6, 8, 10, then will $2+10=4+8=2\times6=12$.

And, if the series be a, $a+d$, $a+2d$, $a+3d$, $a+4d$, then will $a+(a+4d)=(a+d)+(a+3d)=2\times(a+2d)$.

5. The last term of any increasing arithmetical series is equal to the first term *plus* the product of the common difference by the number of terms less one; and if the series be decreasing, it will be equal to the first term *minus* that product.

Thus, the nth or last term (l), of the series a, $a+d$, $a+2d$, $a+3d$, $a+4d$, *&c.*, is $a+(n-1)d$, or $l=a+(n-1)d$.

And the nth or last term (l), of the series a, $a-d$, $a-2d$, $a-3d$, $a-4d$, *&c.*, is $a-(n-1)d$, or $l=a-(n-1)d$.

6. The sum of any series of quantities in arithmetical progression is equal to the sum of the two extremes multiplied by half the number of terms.

Thus, the sum of 2, 4, 6, 8, 10, 12, is $=(2+12)\times\frac{6}{2}=(2+12)\times3=14\times3=42$.

OF

ARITHMETICAL PROPORTION AND PROGRESSION.

Arithmetical Proportion, is the relation which two numbers, or quantities, of the same kind, have to two others, when the difference of the first pair is equal to that of the second.

Hence, three quantities are in arithmetical proportion, when the difference of the first and second is equal to the difference of the second and third.

Thus, 2, 4, 6, and a, $a+b$, $a+2b$, are quantities in arithmetical proportion.

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Thus, 1, 3, 5, 7, 9, *&c.*, and a, $a+d$, $a+2d$, $a+3d$, *&c.*, are increasing series in arithmetical progression, the common differences of which are 2 and d.

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The most useful properties of arithmetical proportion and progression are contained in the following theorems:

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2. And if three quantities be in arithmetical propor-

tion, the sum of the two extremes will be double the mean.

Thus, if the proportionals be 3, 6, 9, or a, b, c, then will $3+9=2\times 6=12$, and $a+c=2b$.

3. Hence an arithmetical mean between any two quantities is equal to half the sum of those quantities.

Thus, an arithmetical mean between 2 and 4 is $=\frac{2+4}{2}=3$; and between 5 and 6 it is $=\frac{5+6}{2}=5\frac{1}{2}$.

And an arithmetical mean between a and b is $\frac{a+b}{2}$.

4. In any continued arithmetical progression, the sum of the two extremes is equal to the sum of any two terms that are equally distant from them, or to double the middle term, when the number of terms is odd.

Thus, if the series be 2, 4, 6, 8, 10, then will $2+10=4+8=2\times 6=12$.

And, if the series be a, $a+d$, $a+2d$, $a+3d$, $a+4d$, then will $a+(a+4d)=(a+d)+(a+3d)=2\times(a+2d)$.

5. The last term of any increasing arithmetical series is equal to the first term *plus* the product of the common difference by the number of terms less one; and if the series be decreasing, it will be equal to the first term *minus* that product.

Thus, the nth or last term (l), of the series a, $a+d$, $a+2d$, $a+3d$, $a+4d$, &c., is $a+(n-1)d$, or $l=a+(n-1)d$.

And the nth or last term (l), of the series a, $a-d$, $a-2d$, $a-3d$, $a-4d$, &c., is $a-(n-1)d$, or $l=a-(n-1)d$.

6. The sum of any series of quantities in arithmetical progression is equal to the sum of the two extremes multiplied by half the number of terms.

Thus, the sum of 2, 4, 6, 8, 10, 12, is $=(2+12)\times\frac{6}{2}=(2+12)\times 3=14\times 3=42$.

6. Required the 365th term of the series of even numbers 2, 4, 6, 8, 10, 12, *&c.* Ans. 730.

7. The first term of a decreasing arithmetical series is 10, the common difference $\frac{1}{3}$, and the number of terms 21; required the sum of the series. Ans. 140.

$\frac{n(n+1)}{2}$ 8. One hundred stones being placed on the ground, in a straight line, at the distance of a yard from each other; how far will a person travel, who shall bring them one by one, to a basket, placed at a distance of a yard from the first stone? Ans. 5 miles and 1300 yards.

10^2+100

10100

$10)10100 = 5$ miles " 1300. $a=1+1=2.\ d=2.\ n=100$

OF

GEOMETRICAL PROPORTION

AND

PROGRESSION.

Geometrical Proportion, is the relation which two numbers, or quantities, of the same kind have to two others, when the antecedents, or leading terms of each pair, are the same parts of their consequents, or the consequents of the antecedents.

And if two quantities only are to be compared together, the part, or parts, which the antecedent is of its consequent, or the consequent of the antecedent, is called the ratio; observing, in both cases, always to follow the same method.

Hence, three quantities are in geometrical proportion, when the first is the same part, or multiple, of the second, as the second is of the third.

Thus, 3, 6, 12, and a, ar, ar^2, are quantities in geometrical proportion.

And four quantities are in geometrical proportion, when the first is the same part, or multiple, of the second, as the third is of the fourth.

8. $a=1+1=2:\ n=100:\ l=100+100=2$[illegible]

$S=(a+l)\frac{n}{2}=(2+200)50=10100$ yards

Thus, 2, 8, 3, 12, and a, ar, b, br, are geometrical proportionals.

Direct proportion, is when the same relation subsists between the first of four terms and the second, as between the third and fourth.

Thus, 3, 6, 5, 10, and a, ar, b, br, are in direct proportion.

Inverse, or reciprocal proportion, is when the first and second of four quantities are directly proportional to the reciprocals of the third and fourth.

Thus, 2, 6, 9, 3, and a, ar, br, b, are inversely proportional; because 2, 6, $\frac{1}{9}$, $\frac{1}{3}$, and a, ar, $\frac{1}{br}$, $\frac{1}{b}$, are directly proportional.

Geometrical Progression is when a series of numbers, or quantities, have the same constant ratio; or which increase, or decrease, by a common multiplier, or divisor.

Thus, 2, 4, 8, 16, 32, 64, *&c.*, and a, ar, ar^2, ar^3, ar^4, *&c.*, are series in geometrical progression.

The most useful properties of geometrical proportion and progression are contained in the following theorems:

1. If three quantities be in geometrical proportion, the product of the two extremes will be equal to the square of the mean.

Thus, if the proportionals be 2, 4, 8, or a, b, c, then will $2 \times 8 = 4^2$ and $a \times c = b^2$.

2. Hence, a geometrical mean proportional, between any two quantities, is equal to the square root of their product.

Thus, a geometric mean between 4 and 9 is $= \sqrt{36} = 6$; and between $\frac{1}{2}$ and $\frac{1}{8}$, it is $= \sqrt{\frac{1}{16}}$, or $\frac{1}{4}$.

Also, a geometric mean between a and b is $= \sqrt{ab}$.

3. If four quantities be in geometrical proportion, the product of the two extremes will be equal to that of the means.

Thus, if the proportionals be 2, 4, 6, 12, or a, b, c, d; then will $2 \times 12 = 4 \times 6$, and $a \times d = b \times c$.

4. Hence, the product of the means of four proportional quantities, divided by either of the extremes, will give the other extreme; and the product of the extremes, divided by either of the means, will give the other mean.

Thus, if the proportionals be 3, 9, 5, 15, or a, b, c, d; then will $\frac{9 \times 5}{3} = 15$, and $\frac{3 \times 15}{5} = 9$.

And, $\frac{b \times c}{a} = d$, and $\frac{a \times d}{c} = b$.

5. Also, if any two products be equal to each other, either of the terms of one of them, will be to either of the terms of the other, as the remaining term of the last is to the remaining term of the first.

Thus, if $ad = bc$, or $2 \times 15 = 6 \times 5$, then will any of the following forms of these quantities be proportional:

Directly, $a : b :: c : d$, or $2 : 6 :: 5 : 15$.

Invertedly, $b : a :: d : c$, or $6 : 2 :: 15 : 5$.

Alternately, $a : c :: b : d$, or $2 : 5 :: 6 : 15$.

Conjunctly, $a : a + b :: c : c + d$, or $2 : 8 :: 5 : 20$.

Disjunctly, $a : b \sim a :: c : d \sim c$, or $2 : 4 :: 5 : 10$.

Mixedly, $b + a : b \sim a :: d + c : d \sim c$, or $8 : 4 :: 20 : 10$.

In all of which cases, the product of the two extremes is equal to that of the two means.

6. In any continued geometrical series, the product of the two extremes is equal to the product of any two means that are equally distant from them; or to the square of the mean, when the number of terms is odd.

Thus, if the series be 2, 4, 8, 16, 32; then will

$$2 \times 32 = 4 \times 16 = 8^2.$$

7. In any geometrical series, the last term is equal to the product arising from multiplying the first term by such a power of the ratio as is denoted by the number of terms less one.

Thus, in the series 2, 6, 18, 54, 162, where 3 is the ratio, we shall have $2 \times 3^4 = 2 \times 81 = 162$.

And in the series a, ar, ar^2, ar^3, ar^4, *&c.*, continued to n terms, where r is the ratio, the nth, or last term (l) will be

$$l = ar^{n-1}.$$

8. The sum of any series of quantities in geometrical progression, either increasing or decreasing, is found by multiplying the last term by the ratio, and then dividing the difference of this product and the first term by the difference between the ratio and unity.

Thus, in the series 2, 4, 8, 16, 32, 64, 128, 256, 512, where 2 is the ratio, we shall have $\frac{512 \times 2 - 2}{2-1} = 1024 - 2 = 1022$, the sum of the terms.

Or a similar rule, without considering the last term, may be expressed thus:

Find such a power of the ratio as is denoted by the number of terms of the series; then divide the difference between this power and unity, by the difference between the ratio and unity, and the result, multiplied by the first term, will be the sum of the series.

Thus, in the series $a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1}$, continued to n terms, we shall have

$$S = a\left(\frac{r^n - 1}{r - 1}\right)$$

And if the ratio, or common multiplier, r, in this last series, be a proper fraction, and consequently the series a decreasing one, we shall have, in that case,

$$a + ar + ar^2 + ar^3 + ar^4 +, \text{ \&c., } ad\ infinitum = \frac{a}{1-r}.$$

EXAMPLES.

1. The first term of a geometrical series is 1, the ratio 2, and the number of terms 10; what is the sum of the series?

Here $1 \times 2^9 = 1 \times 512 = 512$, the last term.

And $\dfrac{512 \times 2 - 1}{2 - 1} = \dfrac{1024 - 1}{1} = 1023$, the sum required.

2. The first term of a geometrical series is $\frac{1}{2}$, the ratio $\frac{1}{3}$, and the number of terms 5; required the sum of the series.

Here $\dfrac{1}{2} \times \left(\dfrac{1}{3}\right)^4 = \dfrac{1}{2} \times \dfrac{1}{81} = \dfrac{1}{162}$, the last term.

And $\dfrac{\frac{1}{2} - \frac{1}{162} \times \frac{1}{3}}{1 - \frac{1}{3}} = \dfrac{\frac{1}{2} - \frac{1}{486}}{\frac{2}{3}} = \dfrac{121}{243} \times \dfrac{3}{2} = \dfrac{121}{162}$, the sum.

Or $\frac{1}{2}\left\{\dfrac{1 - (\frac{1}{3})^5}{1 - \frac{1}{3}}\right\} = \frac{1}{2}\left\{\dfrac{3 - (\frac{1}{3})^4}{3 - 1}\right\} = \frac{1}{2}\left(\dfrac{243 - 1}{162}\right) = \dfrac{121}{162}$.

3. Required the sum of 1, 2, 4, 8, 16, 32, &c., continued to 20 terms. Ans. 1048575.

4. Required the sum of $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$, &c., continued to 8 terms.

Ans. $1\frac{127}{128}$

5. Required the sum of $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}$, &c., continued to 10 terms.

Ans. $1\frac{6561}{17122}$

6. A person being asked to dispose of a fine horse, said he would sell him on condition of having a farthing for the first nail in his shoes, a halfpenny for the second, a penny for the third, twopence for the fourth, and so on, doubling the price of every nail, to 32, the number of nails in his four shoes; what would the horse be sold for at that rate? Ans. 4473924*l.* 5*s.* $3\frac{3}{4}$*d.*

OF EQUATIONS.

THE DOCTRINE OF EQUATIONS is that branch of algebra, which treats of the methods of determining the values of unknown quantities by means of their relations to others which are known.

This is done by making certain algebraic expressions equal to each other (which formula, in that case, is called an equation,) and then working by the rules of the art, till the quantity sought is found equal to some given quantity, and consequently becomes known.

The terms of an equation are the quantities of which it is composed; and the parts that stand on the right and left of the sign $=$, are called the two members, or sides, of the equation.

Thus, if $x=a+b$, the terms are x, a, and b; and the meaning of the expression is, that some quantity, x, standing on the left hand side of the equation, is equal to the sum of the quantities a and b on the right hand side.

A simple equation is that which contains only the first power of the unknown quantity: as,

$$x+a=3b, \text{ or } ax=bc, \text{ or } 2x+3a^2=5b^2;$$

where x denotes the unknown quantity, and the other letters, or numbers, the known quantities.

A compound equation is that which contains two or more different powers of the unknown quantity; as,

$$x^2+ax=b, \text{ or } x^3-4x^2+3x=25.$$

Equations are also divided into different orders, or receive particular names, according to the highest power of the unknown quantity contained in any one of their terms: as, quadratic equations, cubic equations, biquadratic equations, *&c.*

Thus, a quadratic equation is that in which the unknown quantity is of two dimensions, or which rises to the second power: as,

$$x^2=20; \; x^2+ax=b, \text{ or } 3x^2+10x=100.$$

A cubic equation is that in which the unknown quantity is of three dimensions, or which rises to the third power: as,

$$x^3=27\ ;\ 2x^3-3x=35\ ;\ \text{or } x^3-ax^2+bx=c.$$

A biquadratic equation is that in which the unknown quantity is of four dimensions, or which rises to the fourth power: as,

$$x^4=25\ ;\ 5x^4-4x=6\ ;\ \text{or } x^4-ax^3+bx^2-cx=d.$$

And so on, for equations of the 5th, 6th, and other higher orders, which are all denominated according to the highest power of the unknown quantity contained in any one of their terms.

The root of an equation is such a number or quantity, as, being substituted for the unknown quantity, will make both sides of the equation vanish, or become equal to each other.

A simple equation can have only one root; but every compound equation has as many roots as it contains dimensions, or as is denoted by the index of the highest power of the unknown quantity, in that equation.

Thus, in the quadratic equation $x^2+2x=15$, the root, or value of x, is either $+3$ or -5; and, in the cubic equation $x^3-9x^2+26x=24$, the roots are 2, 3, and 4, as will be found by substituting each of these numbers for x.

In an equation of an odd number of dimensions, one of its roots will always be real; whereas in an equation of an even number of dimensions, all its roots may be imaginary; as roots of this kind always enter into an equation by pairs.

Such are the equations $x^2-6x+14=0$, and $x^4-2x^3-9x^2+10x+50=0$. (z)

(z) To the properties of equations abovementioned, we may here farther add,

1. That the sum of all the roots of any equation is equal to the coefficient of the second term of that equation, with its sign changed.

2. The

OF THE
RESOLUTION OF SIMPLE EQUATIONS,
Containing only one unknown Quantity.

The resolution of simple, as well as of other equations, is the disengaging the unknown quantity, in all such expressions, from the other quantities with which it is connected, and making it stand alone, on one side of the equation, so as to be equal to such as are known on the other side; for the performing of which, several axioms and processes are required, the most useful and necessary of which are the following: (*a*)

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2. The sum of the products of every two of the roots, is equal to the coefficient of the third term, without any change in its sign.

3. The sum of the products of every three of the roots, is equal to the coefficient of the fourth term, with its sign changed.

4. And so on, to the last, or absolute term, which is equal to the product of all the roots, with the sign changed or not, according as the equation is of an odd or an even number of dimensions.

See, for a more particular account of the general theory of equations, Vol. II. of my *Treatise on Algebra*, 8vo. 2d Ed. 1820.

(*a*) The operations required, for the purpose here mentioned, are chiefly such as are derived from the following simple and evident principles:

1. If the same quantity be added to, or subtracted from, each of two equal quantities, the results will still be equal; which is the same, in effect, as taking any quantity from one side of an equation, and placing it on the other side, with a contrary sign.

2. If all the terms of any two equal quantities be multiplied or divided by the same quantity, the products, or quotients thence arising, will be equal.

3. If two quantities, either simple or compound, be equal to each other, any like powers, or roots, of them will also be equal.

Each of which axioms, and others of a similar kind, will be found sufficiently illustrated, by the processes arising out of the several examples annexed to the six different cases given in the text; as well as in other parts of the present performance.

CASE I.

Any quantity may be transposed from one side of an equation to the other, by changing its sign; and the two members, or sides, will still be equal.

Thus, if $x+3=7$; then will $x=7-3$, or $x=4$.

And, if $x-4+6=8$; then will $x=8+4-6=6$.

Also, if $x-a+b=c-d$; then will $x=a-b+c-d$.

And, if $4x-8=3x+20$; then $4x-3x=20+8$, and consequently $x=28$.

From this rule it also follows, that if a quantity be found on each side of an equation, with the same sign, it may be left out of both of them; and that the signs of all the terms of any equation may be changed from $+$ to $-$, or from $-$ to $+$, without altering its value.

Thus, if $x+5=7+5$; then, by cancelling, $x=7$.

And, if $a-x=b-c$; then, by changing the signs, $x-a=c-b$, or $x=a+c-b$.

CASE II.

If the unknown quantity, in any equation, be multiplied by any number, or quantity, the multiplier may be taken away, by dividing all the rest of the terms by it; and if it be divided by any number, the divisor may be taken away, by multiplying all the other terms by it.

Thus, if $ax=3ab-c$; then will $x=3b-\frac{c}{a}$.

And if $2x+4=16$; then will $x+2=8$,
or $x=8-2=6$.

Also, if $\frac{x}{2}=5+3$; then will $x=10+6=16$.

And if $\frac{2x}{3}-2=4$; then $2x-6=12$, or, by division, $x-3=9$, or $x=9$.

CASE III.

Any equation may be cleared of fractions, by multiplying each of its terms, successively, by the denominators of those fractions; or by multiplying both sides by the product of all the denominators, or by any quantity that is a multiple of them.

Thus, if $\frac{x}{3}+\frac{x}{4}=5$, then, multiplying by 3, we have $x+\frac{3x}{4}=15$; and this, multiplied by 4, gives $4x+3x=60$; whence by addition, $7x=60$, or $x=\frac{60}{7}=8\frac{4}{7}$.

And, if $\frac{x}{4}+\frac{x}{6}=10$; then, multiplying by 12 (which is a multiple of 4 and 6,) $3x+2x=120$, or $5x=120$, or $x=\frac{120}{5}=24$.

It also appears, from this rule, that if the same number, or quantity, be found in each of the terms of an equation, either as a multiplier or divisor, it may be expunged from all of them, without altering the result.

Thus, if $ax=ab+ac$; then, by cancelling, $x=b+c$.

And, if $\frac{x}{a}+\frac{b}{a}=\frac{c}{a}$; then $x+b=c$, or $x=c-b$.

CASE IV.

If the unknown quantity, in any equation, be in the form of a surd, transpose the terms so that this may stand alone, on one side of the equation, and the remaining terms on the other (by Case I.); then involve each of the sides to such a power as corresponds with

the index of the surd, and the equation will be rendered free from any irrational expression.

Thus, if $\sqrt{x}-2=3$; then will $\sqrt{x}=3+2=5$, or, by squaring, $x=5^2=25$.

And, if $\sqrt{(3x+4)}=5$; then will $3x+4=25$, or $3x=25-4=21$, and consequently $x=\frac{21}{3}=7$.

Also, if $\sqrt[3]{(2x+3)}+4=8$; then will $\sqrt[3]{(2x+3)}=8-4=4$, or $2x+3=4^3=64$, and consequently $2x=64-3=61$, or $x=\frac{61}{2}=30\frac{1}{2}$.

CASE V.

If that side of the equation, which contains the unknown quantity, be a complete power, the equation may be reduced to a lower dimension, by extracting the root of the said power on both sides of the equation.

Thus, if $x^2=81$; then $x=\sqrt{81}=9$; and if $x^3=27$, then $x=\sqrt[3]{27}=3$.

Also, if $3x^2-9=24$; then $3x^2=24+9=33$, or $x^2=\frac{33}{3}=11$, and consequently $x=\sqrt{11}$.

And, if $x^2+6x+9=27$; then, since the left hand side of the equation is a complete square, we shall have, by extracting the roots, $x+3=\sqrt{27}=\sqrt{(9\times3)}=3\sqrt{3}$, or $x=3\sqrt{3}-3$.

CASE VI.

Any analogy, or proportion, may be converted into an equation, by making the product of the two extreme terms equal to that of the two means.

Thus, if $3x : 16 :: 5 : 6$; then $3x\times6=16\times5$, or $18x=80$, or $x=\frac{80}{18}=\frac{40}{9}=4\frac{4}{9}$.

And, if $\frac{2x}{3} : a :: b : c$; then will $\frac{2cx}{3} = ab$, or $2cx = 3ab$; or, by division, $x = \frac{3ab}{2c}$.

Also, if $12 - x : \frac{x}{2} :: 4 : 1$; then $12 - x = \frac{4x}{2} = 2x$, or $2x + x = 12$; and consequently $x = \frac{12}{3} = 4$.

MISCELLANEOUS EXAMPLES.

1. Given $5x - 15 = 2x + 6$, to find the value of x.

Here $5x - 2x = 6 + 15$, or $3x = 6 + 15 = 21$; and therefore, by division, we shall have $x = \frac{21}{3} = 7$ Ans.

2. Given $40 - 6x - 16 = 120 - 14x$, to find the value of x.

Here $14x - 6x = 120 - 40 + 16$; or $8x = 136 - 40 = 96$; and therefore, by division, $x = \frac{96}{8} = 12$ Ans.

3. Given $3x^2 - 10x = 8x + x^2$, to find the value of x.

Here $3x - 10 = 8 + x$, by dividing by x; or $3x - x = 8 + 10 = 18$, by transposition.

And consequently $2x = 18$, or by division, $x = \frac{18}{2} = 9$ Ans.

4. Given $6ax^3 - 12abx^2 = 3ax^3 + 6ax^2$, to find the value of x.

Here $2x - 4b = x + 2$, by dividing by $3ax^2$; or $2x - x = 2 + 4b$; and therefore $x = 4b + 2$. Ans.

5. Given $x^2 + 2x + 1 = 16$, to find the value of x.

Here $x + 1 = 4$, by extracting the square root of each side.

And therefore, by transposition, $x = 4 - 1 = 3$. Ans.

6. Given $5ax - 3b = 2dx + c$, to find the value of x.

Here $5ax-2dx=c+3b$; or $(5a-2d)x=c+3b$; and therefore, by division, $x=\frac{c+3b}{5a-2d}$ Ans.

7. Given $\frac{x}{2}-\frac{x}{3}+\frac{x}{4}=10$, to find the value of x.

Here $x-\frac{2x}{3}+\frac{2x}{4}=20$; and $3x-2x+\frac{6x}{4}=60$; or $12x-8x+6x=240$; whence $10x=240$, or $x=24$. Ans.

8. Given $\frac{x-3}{2}+\frac{x}{3}=20-\frac{x-19}{2}$, to find the value of x.

Here $x-3+\frac{2x}{3}=40-x+19$; or $3x-9+2x=120-3x+57$; whence $3x+2x+3x=120+57+9$; that is $8x=186$, or $x=23\frac{1}{4}$ Ans.

9. Given $\sqrt{\frac{2x}{3}}+5=7$, to find the value of x.

Here $\sqrt{\frac{2x}{3}}=7-5=2$; whence by squaring, $\frac{2x}{3}=2^2=4$, and $2x=12$, or $x=6$. Ans.

10. Given $x+\sqrt{(a^2+x^2)}=\frac{2a^2}{\sqrt{(a^2+x^2)}}$, to find the value of x.

Here $x\sqrt{(a^2+x^2)}+a^2+x^2=2a^2$; or $x\sqrt{(a^2+x^2)}=a^2-x^2$, and $x^2(a^2+x^2)=a^4-2a^2x^2+x^4$; whence $a^2x^2+x^4=a^4-2a^2x^2+x^4$, and $a^2x^2=a^4-2a^2x^2$.

Therefore $3a^2x^2=a^4$, or $x^2=\frac{a^4}{3a^2}=\frac{a^2}{3}$; and consequently $x=\sqrt{\frac{a^2}{3}}=a\sqrt{\frac{1}{3}}=a\sqrt{\frac{3}{9}}=\frac{a}{3}\sqrt{3}$, the answer.

EXAMPLES FOR PRACTICE.

1. Given $5x+22-2x=31$, to find the value of x.
Ans. $x=3$

2. Given $4-9y=14-11y$, to find the value of y.
Ans. $y=5$

3. Given $x+18=3x-5$, to find the value of x.
Ans. $x=11\frac{1}{2}$

4. Given $x+\frac{x}{2}+\frac{x}{3}=11$, to determine the value of x.
Ans. $x=6$

5. Given $2x-\frac{x}{2}+1=5x-2$, to find the value of x.
Ans. $x=\frac{6}{7}$

6. Given $\frac{x}{2}+\frac{x}{3}-\frac{x}{4}=\frac{7}{10}$, to determine the value of x.
Ans. $x=1\frac{1}{5}$

7. Given $\frac{x+3}{2}+\frac{x}{3}=4-\frac{x-5}{4}$, to find the value of x.
Ans. $x=3\frac{6}{13}$

8. Given $2+\sqrt{3x}=\sqrt{(4+5x)}$, to find the value of x.
Ans. $x=12$

9. Given $x+a=\frac{x^2}{a+x}$, to find the value of x.
Ans. $x=-\frac{a}{2}$

10. Given $\sqrt{x}+\sqrt{(a+x)}=\frac{2a}{\sqrt{(a+x)}}$, to find the value of x.
Ans. $x=\frac{a}{3}$

11. Given $\frac{ax-b}{4}+\frac{a}{3}=\frac{bx}{2}-\frac{bx-a}{3}$, to find the value of x.
Ans. $x=\frac{3b}{3a-2b}$

12. Given $\sqrt{(a^2+x^2)}=\sqrt[4]{(b^4+x^4)}$, to find the value of x. Ans. $x=\sqrt{\dfrac{b^4-a^4}{2a^2}}$

13. Given $\sqrt{(a+x)}+\sqrt{(a-x)}=\sqrt{ax}$, to find the value of x. Ans. $x=\dfrac{4a^2}{a^2+4}$

14. Given $\dfrac{a}{1+x}+\dfrac{a}{1-x}=b$, to determine the value of x. Ans. $x=\sqrt{\dfrac{b-2a}{b}}$

15. Given $a+x=\sqrt{\{a^2+x\sqrt{(b^2+x^2)}\}}$, to find the value of x. Ans. $x=\dfrac{b^2}{4a}-a$

16. Given $\frac{1}{2}\sqrt{(x^2+3a^2)}-\frac{1}{2}\sqrt{(x^2-3a^2)}=x\sqrt{a}$, to find the value of x. Ans. $x=\sqrt[4]{\dfrac{9a^3}{4-4a}}$

17. Given $\sqrt{(a+x)}+\sqrt{(a-x)}=b$, to find the value of x. Ans. $x=\dfrac{b}{2}\sqrt{(4a-b^2)}$

18. Given $\sqrt[3]{(a+x)}+\sqrt[3]{(a-x)}=b$, to find the value of x. Ans. $x=\sqrt{\left\{a^2-\left(\dfrac{b^3-2a^3}{3b}\right)\right\}}$

19. Given $\sqrt{a}+\sqrt{x}=\sqrt{ax}$, to find the value of x.
Ans. $x=\dfrac{a}{(\sqrt{a}-1)^2}$

20. Given $\sqrt{\dfrac{x+1}{x-1}}+\sqrt{\dfrac{x-1}{x+1}}=a$, to determine the value of x. Ans. $x=\dfrac{a}{\sqrt{a^2-4}}$

21. Given $\sqrt{(a^2+ax)}=a-\sqrt{(a^2-ax)}$, to find the value of x.

Ans. $x=\frac{a}{2}\sqrt{3}$.

22. Given $\sqrt{(a^2-x^2)}+x\sqrt{(a^2-1)}=a^2\sqrt{(1-x^2)}$, to find the value of x.

Ans. $x=\left(\frac{a^2-1}{a^2+3}\right)^{\frac{1}{2}}$

23. Given $\sqrt{(x+a)}=c-\sqrt{(x+b)}$, to find the value of x.

Ans. $x=\left(\frac{c^2+b-a}{2c}\right)^2-b$

24. Given $\sqrt{\frac{b}{a+x}}+\sqrt{\frac{c}{a-x}}=\sqrt[4]{\frac{4bc}{a^2-x^2}}$, to find the value of x.

Ans. $x=a\left(\frac{b+c}{b-c}\right)$

Of the resolution of simple equations, containing two unknown quantities.

When there are two unknown quantities, and two independent simple equations involving them, they may be reduced to one, by any of the three following rules:

RULE I.

Observe which of the unknown quantities is the least involved, and find its value in each of the equations, as if the other was known, by the methods already explained; then let the two values, thus found, be put equal to each other, and there will arise a new equation with only one unknown quantity in it, the value of which may be found as before. (*b*)

(*b*) This rule depends upon the well known axiom, that things which are equal to the same thing, are equal to each other; and the two following methods are founded on principles which are equally simple and obvious.

EXAMPLES.

1. Given $\left\{\begin{array}{l} 2x+3y=23 \\ 5x-2y=10 \end{array}\right\}$ to find the values of x and y.

Here, from the first equation, $x=\frac{23-3y}{2}$,

And from the second equation $x=\frac{10+2y}{5}$,

Whence by equality, we have $\frac{23-3y}{2}=\frac{10+2y}{5}$,

Or $115-15y=20+4y$, or $19y=115-20=95$,

That is, $y=\frac{95}{19}=5$, and $x=\frac{23-15}{2}=4$.

2. Given $\left\{\begin{array}{l} x+y=a \\ x-y=b \end{array}\right\}$ to find the values of x and y.

Here, from the first equation, $x=a-y$,

And from the second, $x=b+y$.

Whence $a-y=b+y$, or $2y=a-b$,

And therefore $y=\frac{a-b}{2}$, and $x=a-y$,

Or, by substitution, $x=a-\frac{a-b}{2},=\frac{a+b}{2}$.

3. Given $\left\{\begin{array}{l} \frac{1}{2}x+\frac{1}{3}y=7 \\ \frac{1}{3}x+\frac{1}{2}y=8 \end{array}\right\}$ to find the values of x and y.

Here, from the first equation, $x=14-\frac{2y}{3}$,

And from the second, $x=24-\frac{3y}{2}$,

Therefore, by equality, $14-\frac{2y}{3}=24-\frac{3y}{2}$,

And consequently $42-2y=72-\frac{9y}{2}$,

Or, by multiplication, $84-4y=144-9y$;

And, therefore, also $5y=144-84=60$,

Or, by division, $y=\frac{60}{5}=12$, and $x=14-\frac{24}{3}=6$.

RULE II.

Find the value of either of the unknown quantities in that equation in which it is the least involved, as in the last rule; then substitute this value in the place of its equal in the other equation, and there will arise a new equation with only one unknown quantity in it; the value of which may be found as before.

EXAMPLES.

1. Given $\left\{\begin{array}{l} x+2y=17 \\ 3x-\ y=2 \end{array}\right\}$ to find the values of x and y.

Here from the first equation, $x=17-2y$; which value, being substituted for x, in the second,

gives $3(17-2y)-y=2$,

Or $51-6y-y=2$, or $7y=51-2=49$,

Whence $y=\frac{49}{7}=7$, and $x=17-2y=3$,

2. Given $\left\{\begin{array}{l} x+y=13 \\ x-y=\ 3 \end{array}\right\}$ to find the values of x and y.

Here from the first equation, $x=13-y$; which value being substituted for x, in the second,

Gives $13-y-y=3$, or $2y=13-3=10$,

Whence $y=\frac{10}{2}=5$, and $x=13-y=8$.

3. Given $\left\{\begin{array}{c} x : y :: a :: b \\ x^2+y^2=c \end{array}\right\}$ to find the values of x and y.

Here the analogy in the first, turned into an equation,

$$\text{gives } bx=ay, \text{ or } x=\frac{ay}{b},$$

And this value, substituted for x in the second,

$$\text{gives}\left(\frac{ay}{b}\right)^2+y^2=c, \text{ or } \frac{a^2y^2}{b^2}+y^2=c,$$

Whence we have $a^2y^2+b^2y^2=b^2c$, or $y^2=\dfrac{b^2c}{a^2+b^2}$

And, consequently, $y=b\sqrt{\dfrac{c}{a^2+b^2}}$, and $x=a\sqrt{\dfrac{c}{a^2+b^2}}$

RULE III.

Let one or both of the given equations be multiplied, or divided, by such numbers, or quantities, as will make the term that contains one of the unknown quantities the same in each of them; then, by adding, or subtracting, the two equations thus obtained, as the case may require, there will arise a new equation, with only one unknown quantity in it, which may be resolved as before.

EXAMPLES.

1. Given $\left\{\begin{array}{c} 3x+5y=40 \\ x+2y=14 \end{array}\right\}$ to find the values of x and y.

First, multiply the second equation by 3, and it will give $3x+6y=42$.

Then, subtract the first equation from this, and it will give $6y-5y=42-40$, or $y=2$.

Whence, also, $x=14-2y=14-4=10$.

2. Given $\left\{\begin{array}{l} 5x-3y=9 \\ 2x+5y=16 \end{array}\right\}$ to find the values of x and y.

Multiply the first equation by 2, and the second by 5; then $10x-6y=18$, and $10x+25y=80$.

And if the former of these be subtracted from the latter there will arise $31y=62$, or $y=\frac{62}{31}=2$.

Whence, by the first equation, $x=\frac{9+3y}{5}=\frac{15}{5}=3$.

EXAMPLES FOR PRACTICE.

1. Given $4x+y=34$, and $4y+x=16$, to find the values of x and y. Ans. $x=8$, $y=2$.

2. Given $2x+3y=16$, and $3x-2y=11$, to find the values of x and y. Ans. $x=5$, $y=2$.

3. Given $\frac{2x}{5}+\frac{3y}{4}=\frac{9}{20}$, and $\frac{3x}{4}+\frac{2y}{5}=\frac{61}{120}$ to find the values of x and y. Ans. $x=\frac{1}{2}$, $y=\frac{1}{3}$.

4. Given $\frac{x}{7}+7y=99$, and $\frac{y}{7}+7x=51$, to find the values of x and y. Ans. $x=7$, $y=14$.

5. Given $\frac{x}{2}-12=\frac{y}{4}+8$, and $\frac{x+y}{5}+\frac{x}{3}-8=\frac{2y-x}{4}+27$, to find the values of x and y. Ans. $x=60$, $y=40$.

6. Given $x+y=s$, and $x^2-y^2=d$, to find the values of x and y. Ans. $x=\frac{s^2+d}{2s}$, $y=\frac{s^2-d}{2s}$.

7. Given $x+y : a :: x-y : b$, and $x^2-y^2=c$, to find the values of x and y.

Ans. $x=\frac{a+b}{2}\sqrt{\frac{c}{ab}}$, $y=\frac{a-b}{2}\sqrt{\frac{c}{ab}}$.

8. Given $ax+by=c$, and $dx+ey=f$, to find the values of x and y.

Ans. $x=\frac{ce-bf}{ae-bd}$, $y=\frac{af-dc}{ae-bd}$.

9. Given $x^2+y^2=a$, and $x^2-y^2=b$, to find the values of x and y.

Ans. $x=\frac{a+b}{\sqrt{2}}$, $y=\sqrt{\frac{a-b}{2}}$.

10. Given $x^2+xy=a$, and $y^2+xy=b$, to find the values of x and y.

Ans. $x=\frac{-a}{\sqrt{(a+b)}}$, $y=\frac{b}{\sqrt{(a+b)}}$.

Of the resolution of simple equations, containing three or more unknown quantities.

When there are three unknown quantities, and three independent simple equations containing them, they may be reduced to one, by the following method.

RULE.

Find the values of one of the unknown quantities, in each of the three given equations, as if all the rest were known; then put the first of these values equal to the second, and either the first or second equal to the third, and there will arise two new equations with only two unknown quantities in them, the values of which may be found as in the former case; and thence the value of the third.

Or, multiply each of the equations by such numbers, or quantities, as will make one of their terms the same in them all; then, having subtracted any two of these resulting equations from the third, or added them together, as the case may require, there will remain only two equations, which may be resolved by the former rules.

And in nearly the same way may four, five, *&c.*, unknown quantities be exterminated from the same num-

ber of independent simple equations; but, in cases of this kind, there are frequently shorter and more commodious methods of operation, which can only be learnt from practice.

EXAMPLES.

1. Given $\left\{\begin{array}{l} x+y+z=29 \\ x+2y+3z=62 \\ \frac{1}{2}x+\frac{1}{3}y+\frac{1}{4}z=10 \end{array}\right\}$ to find x, y, and z,

Here, from the first equation, $x=29-y-z$.

From the second, $x=62-2y-3z$,

And from the third, $x=20-\frac{2}{3}y-\frac{1}{2}z$,

Whence $29-y-z=62-2y-3z$,

And, also, $29-y-z=20-\frac{2}{3}y-\frac{1}{2}z$,

From the first of which $y=33-2z$,

And from the second, $y=27-\frac{3}{2}z$,

Therefore $33-2z=27-\frac{3}{2}z$, or $z=12$,

Whence, also, $y=33-2z=9$

And $x=29-y-z=8$

2. Given $\left\{\begin{array}{l} 2x+4y-3z=22 \\ 4x-2y+5z=18 \\ 6x+7y-z=63 \end{array}\right\}$ to find x, y, and z.

Here, multiplying the first equation by 6, the second by 3, and the third by 2, we shall have

$$12x+24y-18z=132,$$
$$12x-6y+15z=54,$$
$$12x+14y-2z=126.$$

And, subtracting the second of these equations successively from the first and third, there will arise

$$30y-33z=78,$$
$$20y-17z=72,$$

Or, by dividing the first of these two equations by 3, and then multiplying the result by 2,

$$20y-22x=52,$$
$$20y-17z=72.$$

Whence, by subtracting the former of these from the latter, we have $5z=20$, or $z=4$.

And, consequently, by substitution and reduction,

$$y=7 \text{ and } x=3.$$

3. Given $x+y+z=53$, $x+2y+3z=105$, and $x+3y+4z=134$, to find the values of x, y, and z.

Ans. $x=24$, $y=6$, and $z=23$.

4. Given $x+\frac{1}{2}y+\frac{1}{3}z=32$, $\frac{1}{3}x+\frac{1}{4}y+\frac{1}{5}z=15$, and $\frac{1}{4}x+\frac{1}{5}y+\frac{1}{6}z=12$, to find the values of x, y, and z.

Ans. $x=12$, $y=20$, $z=30$.

5. Given $7x+5y+2z=79$, $8x+7y+9z=122$, and $x+4y+5z=55$, to find the values of x, y, and z.

Ans. $x=4$, $y=9$, $z=3$.

6. Given $x+y=a$, $x+z=b$, and $y+z=c$, to find the values of x, y, and z.

MISCELLANEOUS QUESTIONS,

PRODUCING SIMPLE EQUATIONS.

The usual method of resolving algebraic questions, is first to denote the quantities, that are to be found, by x, y, or some of the other final letters of the alphabet; then, having properly examined the state of the question, perform with these letters, and the known quantities, by means of the common signs, the same operations and reasonings, that it would be necessary to make if the quantities were known, and it was required to verify them, and the conclusion will give the result sought.

Or, it is generally best, when it can be done, to denote only one of the unknown quantities by x, y, or z; and then

to determine the expression for the others, from the nature of the question; after which the same method of reasoning may be followed, as above. And, in certain cases, the substituting for the sums and differences of quantities, or other methods of proceeding may be used; which practice and experience alone can suggest.

1. What number is that whose third part exceeds its fourth part by 16?

Let x = the number required,

Then its $\frac{1}{3}$ part will be $\frac{1}{3}x$, and its $\frac{1}{4}$ part $\frac{1}{4}x$.

And therefore $\frac{1}{3}x - \frac{1}{4}x = 16$, by the question,

That is $x - \frac{3}{4}x = 48$, or $4x - 3x = 192$,

Hence $x = 192$, the number required.

2. It is required to find two numbers such, that their sum shall be 40, and their difference 16.

Let x denote the least of the two numbers required,

Then will $x + 16 =$ to the greater number,

And $x + x + 16 = 40$, by the question,

That is, $2x = 40 - 16$, or $x = \frac{24}{2} = 12 =$ least number.

And $x + 16 = 12 + 16 = 28 =$ the greater number required.

3. Divide 1000*l*. between A, B, and C, so that A shall have 72*l*. more than B, and C 100*l*. more than A.

Let $x =$ B's share of the given sum,

Then will $x + 72 =$ A's share,

And $x + 172 =$ C's share,

Hence their sum is $x + x + 72 + x + 172$,

Or $3x + 244 = 1000$, by the question,

That is, $3x = 1000 - 244 = 756$,

Or $x = \frac{756}{3} = 252l. =$ B's share,

Hence $x+72=324l.$ A's share.
And $x+172=424l.$ C's share.
Also, as above, $252l.$ B's share.

Sum of all $=1000l.$ the proof.

4. It is required to divide $1000l.$ between two persons, so that their shares of it shall be in the proportion of 7 to 9.

Let $x=$the first person's share,
Then will $1000-x=$second person's share,
And $x : 1000-x :: 7 : 9$, by the question,
That is, $9x=(1000-x)\times 7=7000-7x$,

Or $9x+7x=7000$, or $x=\frac{7000}{16}=437l.\ 10s.=$ 1st share,

and $1000-x=1000-437l.\ 10s.=562l.\ 10s.=$ 2d share.

5. The paving of a square court with stones, at $2s.$ a yard, will cost as much as the enclosing it with palisades, at $5s.$ a yard; required the side of the square?

Let $x=$length of the side of the square sought,
Then $4x=$number of yards of enclosure,
And $x^2=$number of yards of pavement,
Hence $4x\times 5=20x=$price of enclosing it,
And $x^2\times 2=2x^2=$the price of the paving,
Therefore $2x^2=20x$, by the question,
And consequently $2x=20$, and by division, $x=10$, whence 10 yards$=$length of the required side.

6. Out of a cask of wine, which had leaked away a third part, 21 gallons were afterwards drawn, and the cask being then gauged, appeared to be half full; how much did it hold?

Let $x=$the number of gallons the cask is supposed to have held,
Then it would have leaked away $\frac{1}{3}x$ gallons.

Whence there had been taken out of it, altogether, in wine and water, $21+\frac{1}{3}x$ gallons,

And therefore $21+\frac{1}{3}x=\frac{1}{2}x$, by the question.

That is, $63+x=\frac{3}{2}x$, or $126+2x=3x$,

Consequently $3x-2x=126$, or $x=126$, the number of gallons required.

7. What fraction is that, to the numerator of which if 1 be added, its value will be $\frac{1}{3}$, but if 1 be added to the denominator, its value will be $\frac{1}{4}$.

Let the fraction required be represented by $\frac{x}{y}$,

Then $\frac{x+1}{y}=\frac{1}{3}$, and $\frac{x}{y+1}=\frac{1}{4}$, by the question.

Hence $3x+3=y$, and $4x=y+1$, or $x=\frac{y+1}{4}$,

Therefore $3\left(\frac{y+1}{4}\right)+3=y$, or $3y+3+12=4y$,

That is, $y=15$, and $x=\frac{y+1}{4}=\frac{15+1}{4}=\frac{16}{4}=4$,

Whence the fraction, that was to be found is $\frac{4}{15}$.

8. A market woman bought a certain number of eggs at 2 a penny, and as many others at 3 a penny, and having sold them again, altogether, at the rate of 5 for 2*d*., found she had lost 4*d*.; how many eggs had she?

Let $x=$the number of eggs of each sort,

Then will $\frac{1}{2}x=$ the price of the first sort,

And $\frac{1}{3}x=$ the price of the second sort,

But $\frac{2}{5}$ of the whole number of eggs, or $\frac{2}{5}\times 2x=\frac{4x}{5}$ the price of both sorts, when mixed together, at the rate of 5 for $2d.$,

And consequently $\frac{1}{2}x+\frac{1}{3}x-\frac{4x}{5}=4$, by the question,

That is, $15x+10x-24x=120$, or $x=120$, the number of eggs of each sort, as required.

9. If A can perform a piece of work in 10 days, and B in 13; in what time will they finish it, if they are both set about it together?

Let the time sought be denoted by x,

Then $\frac{x}{10}=$ the part done by A in one day,

And $\frac{x}{13}=$ the part done by B in one day,

Consequently $\frac{x}{10}+\frac{x}{13}=1$ (the whole work)

That is, $13x+10x=130$, or $23x=130$,

Whence $x=\frac{130}{23}=5\frac{15}{23}$ days, the time required.

10. If one agent A, alone, can produce an effect e, in the time a, and another agent B, alone, in the time b; in what time will both of them together produce the same effect?

Let the time sought be denoted by x,

Then $a : e :: x : \frac{ex}{a}=$ part of the effect produced by A,

And $b : e :: x : \frac{ex}{b}=$ part of the effect produced by B,

Hence $\frac{ex}{a}+\frac{ex}{b}=e$ (the whole effect) by the question.

Or $\frac{x}{a}+\frac{x}{b}=1$ by dividing each side by e,

Therefore $x+\frac{ax}{b}=a$, or $bx+ax=ab$,

Consequently $x=\frac{ab}{a+b}=$ the time required.

11. How much rye at 4*s.* 6*d.* a bushel, must be mixed with 50 bushels of wheat, at 6*s.* a bushel, so that the mixture may be worth 5*s.* a bushel?

Let $x=$ the number of bushels required,
Then $9x$ is the price of the rye in sixpences,
And 600 the price of the wheat in ditto,
Also $(50+x)\times 10$ the price of the mixture in ditto,
Whence $9x+600=500+10x$, by the question,
Or, by transposition, $10x-9x=600-500$,
Consequently $x=100$ the number of bushels required.

12. A labourer engaged to serve for 40 days, on condition that for every day he worked he should receive 20*d.*, but for every day he was absent he should forfeit 8*d.*: now, at the end of the time, he had to receive 1*l.* 11*s.* 8*d.*; how many days did he work, and how many was he idle?

Let the number of days that he worked be denoted by x,
Then will $40-x$ be the number of days he was idle,
Also $20x$ the sum earned, and $(40-x)\times 8$,
or $320-8x$ the sum forfeited,
Whence $20x-(320-8x)=380d.(=1l.\ 11s.\ 8d.)$, by the question,

That is, $20x-320+8x=380$,
Or $28x=380+320=700$,

Consequently $x=\frac{700}{28}=25$, the number of days he

worked, and $40-x=40-25=15$, the number of days he was idle.

QUESTIONS FOR PRACTICE.

1. It is required to divide a line, of 15 inches in length, into two such parts, that one of them may be three-fourths of the other. Ans. $8\frac{4}{7}$ and $6\frac{3}{7}$.

2. My purse and money together are worth 20*s.*, and the money is worth 7 times as much as the purse, how much is there in it? Ans. 17*s.* 6*d.*

3. A shepherd, being asked how many sheep he had in his flock, said, if I had as many more, half as many more, and 7 sheep and a half, I should have just 500; how many had he? Ans. 197.

4. A post is one fourth of its length in the mud, one third in the water, and 10 feet above the water; what is its whole length? Ans. 24 feet.

5. After paying away a fourth of my money, and then a fifth of the remainder, I had 72 guineas left; what had I at first? Ans. 120 guineas.

6. It is required to divide 300*l.* between A, B, and C, so that A may have twice as much as B, and C as much as A and B together. Ans. A 100*l.*, B 50*l.*, C 150*l.*

7. A person, at the time he was married, was 3 times as old as his wife; but after they had lived together 15 years, he was only twice as old; what were their ages on their wedding day?

Ans. Bride's age 15, bridegroom's 45.

8. It is required to find a number such, that if 5 be subtracted from it, two thirds of the remainder shall be 40? Ans. 65.

9. At a certain election, 1296 persons voted, and the successful candidate had a majority of 120; how many voted for each?

Ans. 708 for one, and 588 for the other.

10. A's age is double of B's, and B's is triple of C's, and the sum of all their ages is 140; what is the age of each? Ans. A's 84, B's 42, and C's 14.

11. Two persons, A and B, lay out equal sums of money in trade; A gains 126*l*., and B loses 87*l*., and A's money is now double of B's; what did each lay out?

Ans. 300*l*.

12. A person bought a chaise, horse, and harness, for 60*l*.; the horse came to twice the price of the harness, and the chaise to twice the price of the horse and harness; what did he give for each?

Ans. 13*l*. 6*s*. 8*d*. for the horse, 6*l*. 13*s*. 4*d*. for the harness, and 40*l*. for the chaise.

13. A person was desirous of giving 3*d*. apiece to some beggars, but found he had not money enough in his pocket by 8*d*., he therefore gave them each 2*d*., and had then 3*d*. remaining; required the number of beggars?

Ans. 11.

14. A servant agreed to live with his master for 8*l*. a year, and a livery, but was turned away at the end of seven months, and received only 2*l*. 13*s*. 4*d*. and his livery; what was its value? Ans. 4*l*. 16*s*.

15. A person left 560*l*. between his son and daughter, in such a manner, that for every half-crown the son should have, the daughter was to have a shilling; what were their respective shares?

Ans. Son 400*l*., Daughter 160*l*.

16. There is a certain number, consisting of two places of figures, which is equal to four times the sum of its digits; and if 18 be added to it the digits will be inverted; what is the number? Ans. 24.

17. Two persons, A and B, have both the same income; A saves a fifth of his yearly, but B, by spending 50*l*. per annum more than A, at the end of four years, finds himself 100*l*. in debt; what was their income?

Ans. 125*l*.

18. When a company at a tavern came to pay their reckoning, they found, that if there had been three persons more, they would have had a shilling apiece less to pay, and if there had been two less, they would have had a shilling apiece more to pay; required the number of persons, and the quota of each?

Ans. 12 persons, quota of each $5s.$

19. A person at a tavern borrowed as much money as he had about him, and out of the whole spent $1s.$; he then went to a second tavern, where he also borrowed as much as he had now about him, and out of the whole spent $1s.$; and going on, in this manner, to a third and fourth tavern, he found, after spending his shilling at the latter, that he had nothing left, how much money had he at first? Ans. $11\frac{1}{4}d.$

20. It is required to divide the number 75 into two such parts, that three times the greater shall exceed seven times the less by 15. Ans. 54 and 21.

21. In a mixture of British spirits and water, one half of the whole plus 25 gallons was spirits, and a third part minus 5 gallons was water; how many gallons were there of each? Ans. 85 of wine, and 35 of water.

22. A bill of $120l.$ was paid in guineas and moidores, and the number of pieces of both sorts that were used was just 100; how many were there of each, reckoning the guinea at $21s.$, and the moidore at $27s.$? Ans. 50.

23. Two travellers set out at the same time from London and York, whose distance from each other is 197 miles; one of them goes 14 miles a day, and the other 16; in what time will they meet?

Ans. 6 days $13\frac{3}{5}$ hours.

24. There is a fish whose tail weighs $9lb.$, his head weighs as much as his tail and half his body, and his body weighs as much as his head and his tail; what is the whole weight of the fish? Ans. $72lb.$

25. It is required to divide the number 10 into three

such parts, that, if the first be multiplied by 2, the second by 3, and the third by 4, the three products shall be all equal. Ans. $4\frac{8}{13}$, $3\frac{1}{13}$, and $2\frac{4}{13}$.

26. It is required to divide the number 36 into three such parts, that half the first, a third of the second, and a fourth of the third, shall be all equal to each other.

Ans. The parts are 8, 12, and 16.

27. A person has a saddle worth 50*l.* which being put on the back of one of his two best horses, will make his value double that of the other, and if it be put on the back of the latter, it will make his value triple that of the former; what is the value of each horse?

Ans. One 30*l.* and the other 40*l.*

28. A, in playing at billiards with B, won 5*s.* of him, and had then twice as much money as B had left; but B, in winning back his own money and 5*s.* more, had now three times as much as A had left; how much had each at first? Ans. A 11*s.* and B 13*s.*

29. What two numbers are those whose difference, sum, and product, are to each other as the numbers 2, 3, and 5, respectively? Ans. 10 and 2.

30. A person in play lost a fourth of his money, and then won back 3*s.*, after which he lost a third of what he now had, and then won back 2*s.*; lastly, he lost a seventh of what he then had, and after this found he had but 12*s.* remaining; what had he at first? Ans. 20*s.*

31. A hare is 50 leaps before a greyhound, and takes 4 leaps to the greyhound's 3, but 2 of the greyhound's leaps are as much as 3 of the hare's, how many leaps must the greyhound take to catch the hare? Ans. 300.

32. It is required to divide the number 90 into four such parts, that if the first part be increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, the sum, difference, product, and quotient, shall be all equal?

Ans. The parts are 18, 22, 10 and 40.

33. A person after spending 10*l.* more than a third of his yearly income, found he had 15*l.* more than half of it remaining; what was his income? Ans. 150*l.*

34. A man and his wife usually drank out a cask of beer in 12 days, but when the man was from homeit lasted the woman 30 days; how many days would the man alone be in drinking it? Ans. 20 days.

35. A general, ranging a division of his army in the form of a solid square, finds he has 34 men to spare, but increasing the side by one man, he wants 59 to fill up the square; how many soldiers did the division consist of? Ans. 2150.

36. If A and B together can perform a piece of work in 8 days, A and C together in 9 days, and B and C in 10 days, how many days will it take each person to perform the same work alone?

Ans. A $14\frac{34}{49}$ days, B $17\frac{23}{41}$, and C $23\frac{7}{31}$.

QUADRATIC EQUATIONS.

A QUADRATIC EQUATION, as before observed, is that in which the unknown quantity is of two dimensions, or which rises to the second power; and is either simple or compound.

A simple quadratic equation, is that which contains only the square, or second power, of the unknown quantity; as $ax^2=b$, or $x^2=\frac{b}{a}$; where $x=\sqrt{\frac{b}{a}}$.

A compound quadratic equation, is that which contains both the first and second power of the unknown quantity; as $ax^2+bx=c$, or $x^2+\frac{b}{a}x=\frac{c}{a}$.

In which case, it is to be observed, that every equation of this kind, having any real positive root, will fall under one or other of the three following forms:

1. $x^2+ax=b$ . . . where $x=-\frac{a}{2}\pm\sqrt{(\frac{a^2}{4}+b)}$.

2. $x^2-ax=b$ . . . where $x=+\frac{a}{2}\pm\sqrt{(\frac{a^2}{4}+b)}$.

3. $x^2-ax=-b$ . . where $x=+\frac{a}{2}\pm\sqrt{(\frac{a^2}{4}-b)}$.

Or, if the second and last terms be taken either positively or negatively, as they may happen to be, the general equation

$$ax^2\pm bx=\pm c, \text{ or } x^2\pm\frac{b}{a}x=\pm\frac{c}{a}$$

which comprehends all the three cases above mentioned, may be resolved by means of the following rule:

RULE.

Transpose all the terms that involve the unknown quantity to one side of the equation, and the known terms to the other; observing to arrange them so, that the term which contains the square of the unknown quantity may be positive, and stand first in the equation.

Then, if this square has any coefficient prefixed to it, let all the rest of the terms be divided by it, and the equation will be brought to one of the three forms above-mentioned.

In which case, the value of the unknown quantity x is always equal to half the coefficient, or multiplier of x, in the second term of the equation, taken with a contrary sign, together with $\pm$ the square root of the square of this number and the known quantity that forms the absolute, or third term of the equation. (*c*.)

(*c*) This rule, which is more commodious in its practical application, than that usually given, is founded upon the same principle; being derived from the well-known property, that in any quadratic equation

$$x^2\pm ax=\pm b$$

if the square of half the coefficient a, of the second term of the

Note. All equations, which have the index of the unknown quantity, in one of their terms, just double that of the other, are resolved like quadratics, by first finding the value of the square root of the first term, according to the method used in the above rule, and then taking such a root, or power of the result, as is denoted by the reduced index of the unknown quantity.

Thus, if there be taken any general equation of this kind, as

$$x^{2m}+ax^m=b,$$

we shall have, by taking the square root of x^{2m}, and observing the latter part of the rule,

$$x^m=-\frac{a}{2}\pm\sqrt{(\frac{a^2}{4}+b)},\text{and}x=\left\{-\frac{a}{2}\pm\sqrt{(\frac{a^2}{4}+b)}\right\}^{\frac{1}{m}}$$

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equation, be added to each of its sides, so as to render it of the form

$$x^2\pm ax+\tfrac{1}{4}a^2=\tfrac{1}{4}a^2\pm b$$

that side which contains the unknown quantity will then be a complete square; and, consequently, by extracting the root of each side, we shall have

$$x\pm\tfrac{1}{2}a=\pm\sqrt{(\tfrac{1}{4}a^2\pm b)},\text{ or } x=\mp\tfrac{1}{2}a\pm\sqrt{(\tfrac{1}{4}a^2\pm b)}$$

which is the same as the rule, taking a and b in + or − as they may happen to be.

It may here, also, be observed, that the ambiguous sign ±, which denotes both + and −, is prefixed to the radical part of the value of x in every expression of this kind, because the square root of any positive quantity, as a^2, is either $+a$, or $-a$; for $(+a)\times(+a)$, or $(-a)\times(-a)$ are each $=+a^2$: but the square root of a negative quantity, as $-a^2$, is imaginary, or unassignable, there being no quantity, either positive or negative, that, when multiplied by itself, will give a negative product.

To this we may also further add, that from the constant occurrence of the double sign before the radical part of the above expression, it necessarily follows, that every quadratic equation must have two roots; which are either both real, or both imaginary, according to the nature of the question.

And if the equation, which is to be resolved, be of the following form

$$x^m - ax^{\frac{m}{2}} = b,$$

we shall necessarily have, according to the same principle,

$$x^{\frac{m}{2}} = \frac{a}{2} \pm \sqrt{\left(\frac{a^2}{4} + b\right)}, \text{ and } x = \left\{\frac{a}{2} \pm \sqrt{\left(\frac{a^2}{4} + b\right)}\right\}^{\frac{2}{m}}$$

EXAMPLES.

1. Given $x^2 + 4x = 140$, to find the value of x.
 Here $x^2 + 4x = 140$, by the question,
 Whence $x = -2 \pm \sqrt{(4 + 140)}$, by the rule,
Or, which is the same thing, $x = -2 \pm \sqrt{144}$,
Wherefore $x = -2 + 12 = 10$, or $-2 - 12 = -14$,
Where one of the values of x is positive and the other negative.

2. Given $x^2 - 12x + 30 = 3$, to find the value of x.
 Here $x^2 - 12x = 3 - 30 = -27$, by transposition,
 Whence $x = 6 \pm \sqrt{(36 - 27)}$, by the rule,
 Or, which is the same thing, $x = 6 \pm \sqrt{9}$,
 Therefore $x = 6 + 3 = 9$, or $= 6 - 3 = 3$,
 Where it appears that x has two positive values.

3. Given $2x^2 + 8x - 20 = 70$, to find the value of x.
 Here $2x^2 + 8x = 70 + 20 = 90$, by transposition,
 And $x^2 + 4x = 45$, by dividing by 2,
 Whence $x = -2 \pm \sqrt{(4 + 45)}$, by the rule,
 Or, which is the same thing, $x = -2 \pm \sqrt{49}$,
 Therefore $x = -2 + 7 = 5$, or $= -2 - 7 = -9$,
Where one of the values of x is positive and the other negative.

4. Given $3x^2 - 3x + 6 = 5\frac{1}{3}$, to find the value of x.

 Here $3x^2 - 3x = 5\frac{1}{3} - 6 = -\frac{2}{3}$ by transposition.

 And $x^2 - x = -\frac{2}{9}$ by dividing by 3,

Whence $x=\frac{1}{2}\pm\sqrt{\left(\frac{1}{4}-\frac{2}{9}\right)}$, by the rule,

Or, by subtracting $\frac{2}{9}$ from $\frac{1}{4}$, $x=\frac{1}{2}\pm\sqrt{\frac{1}{36}}$,

Therefore $x=\frac{1}{2}+\frac{1}{6}=\frac{2}{3}$, or $=\frac{1}{2}-\frac{1}{6}=\frac{1}{3}$,

In which case x has two positive values.

5. Given $\frac{1}{2}x^2-\frac{1}{3}x+20\frac{1}{2}=42\frac{2}{3}$ to find the value of x.

Here $\frac{1}{2}x^2-\frac{1}{3}x=42\frac{2}{3}-20\frac{1}{2}=22\frac{1}{6}$ by transposition,

And $x^2-\frac{2}{3}x=44\frac{1}{3}$, by dividing by $\frac{1}{2}$, or multiplying by 2,

Whence we have $x=\frac{1}{3}\pm\sqrt{(\frac{1}{9}+44\frac{1}{3})}$, by the rule,

Or, by adding $\frac{1}{9}$ and $44\frac{1}{3}$ together, $x=\frac{1}{3}\pm\sqrt{\frac{400}{9}}$,

Therefore $x=\frac{1}{3}+6\frac{2}{3}=7$, or $=\frac{1}{3}-6\frac{2}{3}=-6\frac{1}{3}$,

Where one value of x is positive, and the other negative.

6. Given $ax^2+bx=c$, to find the value x.

Here $x^2+\frac{b}{a}x=\frac{c}{a}$ by dividing each side by a.

Whence, by the rule, $x=-\frac{b}{2a}\pm\sqrt{\left(\frac{b^2}{4a^2}+\frac{c}{a}\right)}$,

Or, reducing the part within the radical $x=-\frac{b}{2a}\pm$

$\sqrt{\frac{b^2+4ac}{4a^2}}$,

Therefore $x=-\frac{b}{2a}\pm\frac{1}{2a}\sqrt{(b^2+4ac.)}$

7. Given $ax^2-bx+c=d$, to find the value of x.

Here $ax^2-bx=d-c$, by transposition,

And $x^2-\frac{b}{a}x=\frac{d-c}{a}$, by dividing by a.

Whence $x=\frac{b}{2a}\pm\surd\left(\frac{d-c}{a}+\frac{b^2}{4a^2}\right)$ by the rule,

Or, mult^s. $d-c$ & a by $4a$, $x=\frac{b}{2a}\pm\frac{1}{2a}\surd\{4a(d-c)+b^2\}$

8. Given $x^4+ax^2=b$, to find the value of x.

Here $x^4+ax^2=b$, by the question.

Or, $x^2=-\frac{a}{2}\pm\surd(\frac{a^2}{4}+b)=-\frac{a}{2}\pm\frac{1}{2}\surd(a^2+4b)$, by the rule

Whence $x=\pm\surd\{-\frac{a}{2}\pm\frac{1}{2}\surd(4b+a^2)\}$ by extracting the root on each side.

9. Given $\frac{1}{2}x^6-\frac{1}{4}x^3=-\frac{1}{32}$, to find the value of x.

Here $\frac{1}{2}x^6-\frac{1}{4}x^3=-\frac{1}{32}$, by the question,

And $x^6-\frac{1}{2}x^3=-\frac{1}{16}$, by multiplying by 2,

Whence $x^3=\frac{1}{4}\pm\surd\left(\frac{1}{16}-\frac{1}{16}\right),=\frac{1}{4}$ by the rule,

And consequently $x=\sqrt[3]{\frac{1}{4}}=\sqrt[3]{\frac{2}{8}}=\frac{1}{2}\sqrt[3]{2}$.

10. Given $2x^{\frac{2}{3}}+3x^{\frac{1}{3}}=2$, to find the value of a,

Here $2x^{\frac{2}{3}}+3x^{\frac{1}{3}}=2$, by the question.

And $x^{\frac{2}{3}}+\frac{3}{2}x^{\frac{1}{3}}=1$, by dividing by 2,

Whence $x^{\frac{1}{3}}=-\frac{3}{4}\pm\sqrt{(\frac{9}{16}+1)}=-\frac{3}{4}\pm\frac{5}{4}=\frac{1}{2}$, or -2,

Therefore $x=(\frac{1}{2})^3=\frac{1}{8}$, or $(-2)^3=-8$.

11. Given $x^4-12x^3+44x^2-48x=9009$ (a), to find the value of x.

This equation may be expressed as follows,

$$(x^2-6x)^2+8(x^2-6x)=a,$$

Whence $x^2-6x=-4\pm\sqrt{(16+a)}$, by the common rule,

And, by a 2d operation, $x=3\pm\sqrt{\{9-4\pm\sqrt{(16+a)}\}}$

Therefore, by restoring the value of a, we have

$$x=3\pm\sqrt{(5\pm\sqrt{9025})}$$

Or, by extraction of roots, $x=13$, the Ans.

EXAMPLES FOR PRACTICE. (*d*)

1. Given $x^2-8x+10=19$, to find the value of x.
Ans. $x=9$.

2. Given $x^2-x-40=170$, to find the value of x.
Ans. $x=15$.

3. Given $3x^2+2x-9=76$, to find the value of x.
Ans. $x=5$.

4. Given $\frac{1}{2}x^2-\frac{1}{3}x+7\frac{3}{8}=8$, to find the value of x.
Ans. $x=1\frac{1}{2}$.

(*d*) The unknown quantity in each of the following examples, as well as in those given above, has always two values, as appears from the common rule; but the negative and imaginary roots being, in general, but seldom used in practical questions of this kind, are here suppressed.

5. Given $\frac{1}{2}x - \frac{1}{3}\sqrt{x} = 22\frac{1}{6}$, to find the value of x.
Ans. $x=49$.

6. Given $x+\sqrt{(5x+10)}=8$, to find the value of x.
Ans. $x=3$.

7. Given $\sqrt{(10+x)} - \sqrt[4]{(10+x)} = 2$, to find the value of x.
Ans. $x=6$.

8. Given $2x^4 - x^2 + 96 = 99$, to determine the value of x.
Ans. $x=\frac{1}{2}\sqrt{6}$

9. Given $x^6 + 20x^3 - 10 = 59$, to find the value of x.
Ans. $x=\sqrt[3]{3}$.

10. Given $3x^{2n} - 2x^n + 3 = 11$, to find the value of x.
Ans. $x=\sqrt[n]{2}$.

11. Given $5\sqrt[4]{x} - 3\sqrt{x} = 1\frac{1}{3}$, to determine the value of x.
Ans. $3\frac{13}{81}$ or $\frac{1}{81}$

12. Given $\frac{2}{3}x\sqrt{(3+2x^2)} = \frac{1}{2} + \frac{2}{3}x^2$, to determine the value of x.
Ans. $x=\frac{1}{2}\sqrt{(-3+3\sqrt{2})}$

13. Given $x\sqrt{(\frac{6}{x} - x)} = \frac{1+x^2}{\sqrt{x}}$, to determine the value of x.
Ans. $x=(1+\frac{1}{2}\sqrt{2})^{\frac{1}{2}}$

14. Given $\frac{1}{x}\sqrt{(1-x^3)} = x^2$, to determine the value of x.
Ans. $x=\left(\frac{1}{2}\sqrt{5} - \frac{1}{2}\right)^{\frac{1}{3}}$

15. Given $x\sqrt{(\frac{a}{x} - 1)} = \sqrt{(x^2 - b^2)}$, to determine the value of x.
Ans. $x=\frac{1}{4}a + \frac{1}{4}\sqrt{(8b^2 + a^2)}$

16. Given $\sqrt{1+x-x^2}-2(1+x-x^2)=\frac{1}{9}$, to determine the value of x. Ans. $x=\frac{1}{2}+\frac{1}{6}\sqrt{41}$

17. Given $\sqrt{\left(x-\frac{1}{x}\right)}+\sqrt{\left(1-\frac{1}{x}\right)}=x$, to determine the value of x. Ans. $x=\frac{1}{2}+\frac{1}{2}\sqrt{5}$.

18. Given $x^{4n}-2x^{3n}+x^n=6$, to find the value of x.

Ans. $x=\sqrt[n]{\frac{1}{2}+\frac{1}{2}\sqrt{13}}$.

QUESTIONS PRODUCING QUADRATIC EQUATIONS.

The methods of expressing the conditions of questions of this kind, and the consequent reduction of them, till they are brought to a quadratic equation, involving only one unknown quantity and its square, are the same as those already given for simple equations.

1. To find two numbers such that their difference shall be 8, and their product 240.

Let x equal the least number,
Then will $x+8=$ the greater,
And $x\,(x+8)=x^2+8x=240$, by the question,
Whence $x=-4+\sqrt{(16+240)}=-4+\sqrt{256}$, by the common rule, before given,
Therefore $x=16-4=12$, the less number,
And $x+8=12+8=20$, the greater.

2. It is required to divide the number 60 into two such parts, that their product shall be 864.

Let $x=$ the greater part,
Then will $60-x=$ the less,
And $x\,(60-x)=60x-x^2=864$, by the question,
Or by changing the signs on both sides of the equation
$x^2-60x=-864$.

Whence $x=30\pm\sqrt{(900-864)}=30\pm\sqrt{36}=30\pm6$, by the rule,

And consequently $x=30+6=36$, or $30-6=24$, the two parts sought.

3. It is required to find two numbers such that their sum shall be $10 (a)$, and the sum of their squares $58 (b)$.

Let $x=$the greater of the two numbers,
Then will $a-x=$the less,
And $x^2+(a-x)^2=2x^2-2ax+a^2=b$, by the question,
Or $2x^2-2ax=b-a^2$, by transposition,
And $x^2-ax=\frac{b-a^2}{2}$, by division.

Whence $x=\frac{a}{2}\pm\sqrt{\left(\frac{a^2}{4}+\frac{b-a^2}{2}\right)}=\frac{a}{2}\pm\frac{1}{2}\sqrt{(2b-a^2)}$ by the rule,

And if 10 be put for a, and 58 for b, we shall have

$x=\frac{10}{2}+\frac{1}{2}\sqrt{(116-100)}=7$, the greater number,

And $10-x=\frac{10}{2}-\frac{1}{2}\sqrt{(116-100)}=3$, the less.

4. Having sold a piece of cloth for $24l.$, I gained as much per cent. as it cost me; what was the price of the cloth?

Let $x=$pounds the cloth cost,
Then will $24-x=$the whole gain,
But $100 : x :: x : 24-x$, by the question,
Or $x^2=100(24-x)=2400-100x$,
That is, $x^2+100x=2400$,
Whence $x=-50+\sqrt{(2500+2400)}=-50+70=20$, by the rule,
And consequently $20l.=$price of the cloth.

5. A person bought a number of sheep for 80*l.*, and if he had bought 4 more for the same money, he would have paid 1*l.* less for each; how many did he buy?

Let x represent the number of sheep,

Then will $\frac{80}{x}$ be the price of each,

And $\frac{80}{x+4}=$ price of each, if $x+4$ cost 80*l.*

But $\frac{80}{x}=\frac{80}{x+4}+1$, by the question,

Or $80=\frac{80x}{x+4}+x$, by multiplication,

And $80x+320=80x+x^2+4x$, by the same,

Or, by leaving out $80x$ on each side, $x^2+4x=320$,

Whence $x=-2+\sqrt{(4+320)}=-2+18$, by the rule,

And consequently $x=16$, the number of sheep.

6. It is required to find two numbers, such that the sum, product, and difference of their squares, shall be all equal to each other.

Let $x=$ the greater number, and $y=$ the less.

Then $\left.\begin{cases} x+y=xy \\ x+y=x^2-y^2 \end{cases}\right\}$ by the question,

Hence $1=\frac{x^2-y^2}{x+y}=x-y$, or $x=y+1$, by 2d equation,

And $(y+1)+y=y(y+1)$ by 1st equation,

That is, $2y+1=y^2+y$; or $y^2-y=1$,

Whence $y=\frac{1}{2}+\sqrt{\left(\frac{1}{4}+1\right)}=\frac{1}{2}+\frac{1}{2}\sqrt{5}$, by the rule,

Therefore $y=\frac{1}{2}+\frac{1}{2}\sqrt{5}=1.6180\ldots$

And $x=y+1=\frac{3}{2}+\frac{1}{2}\sqrt{5}=2.6180\ldots$

Where... denotes that the decimal does not end.

7. It is required to find four numbers in arithmetical progression, such that the product of the two extremes shall be 45, and the product of the means 77.

Let x=least extreme, and y=common difference,
Then x, $x+y$, $x+2y$, and $x+3y$, will be the four numbers,

Hence $\left\{ \begin{array}{l} x(x+3y)=x^2+3xy=45 \\ (x+y)(x+2y)=x^2+3xy+2y^2=77 \end{array} \right\}$ by the question,

And $2y^2=77-45=32$, by subtraction,

Or $y^2=\frac{32}{2}=16$ by division, and $y=\sqrt{16}=4$,

Therefore $x^2+3xy=x^2+12x=45$, by the 1st equation,
And consequently $x=-6+\sqrt{(36+45)}=-6+9=3$, by the rule,

Whence the numbers are 3, 7, 11, and 15.

8. It is required to find three numbers in geometrical progression, such that their sum shall be 14, and the sum of their squares 84.

Let x, y, and z be the three numbers,
Then $xz=y^2$, by the nature of proportion,

And $\left\{ \begin{array}{l} x+y+z=14 \\ x^2+y^2+z^2=84 \end{array} \right\}$ by the question,

Hence $x+z=14-y$, by the second equation,
And $x^2+2zx+z^2=196-28y+y^2$, by squaring both sides,
Or $x^2+z^2+2y^2=196-28y+y^2$ by putting $2y^2$ for its equal $2xz$,
That is, $x^2+y^2+z^2=196-28y$ by subtraction,
Or $196-28y=84$ by equality,

Whence $y=\frac{196-84}{28}=4$, by transposition & division,

Again $xz=y^2=16$, or $x=\frac{16}{z}$, by the 1st equation,

And $x+y+z=\frac{16}{z}+4+z=14$, by the 2d equation,

Or $16+4z+z^2=14z$, or $z^2-10z=-16$,

Whence $z=5\pm\sqrt{(25-16)}=5\pm3=8$, or 2 by the rule,

Therefore the three numbers are 2, 4, and 8.

9. It is required to find two numbers, such that their sum shall be 13 (a), and the sum of their fourth powers 4721. (b)

Let $x=$ the difference of the two numbers sought,

Then will $\frac{1}{2}a+\frac{1}{2}x$, or $\frac{a+x}{2}=$ the greater number,

And $\frac{1}{2}a-\frac{1}{2}x$, or $\frac{a-x}{2}=$ the less,

But $\frac{(a+x)^4}{16}+\frac{(a-x)^4}{16}=b$, by the question,

Or $(a+x^4)+(a-x)^4=16b$, by multiplication,

Or $2a^4+12a^2x^2+2x^4=16b$, by involution and addition,

And $x^4+6a^2x^2=8b-a^4$, by transposition and division,

Whence $x^2=-3a^2+\sqrt{(9a^4+8b-a^4)}=-3a^2+\sqrt{8(a^4+b)}$, by the rule,

And $x=\sqrt{\{-3a^2+2\sqrt{2(a^4+b)}\}}$, by extracting the root,

Where, by substituting 13 for a, and 4721 for b, we shall have $x=3$,

Therefore $\frac{13+x}{2}=\frac{16}{2}=8$, the greatest number,

And $\frac{13-x}{2}=\frac{10}{2}=5$, the less number,

The sum of which is 13, and $8^4+5^4=4721$.

QUESTIONS FOR PRACTICE.

1. It is required to divide the number 40 into two such parts, that the sum of their squares shall be 818.
Ans. 23 and 17

2. To find a number such, that if you subtract it from 10, and then multiply the remainder by the number itself, the product shall be 21. Ans. 7 or 3

3. It is required to divide the number 24 into two such parts, that their product shall be equal to 35 times their difference. Ans. 10 and 14

4. It is required to divide the number 20 into two such parts, that twice the square of the greater part shall exceed three times the square of the less, by 96.
Ans. 12 and 8

5. It is required to divide the number 60 into two such parts, that their product shall be to the sum of their squares in the ratio of 2 to 5. Ans. 20 and 40

6. It is required to divide the number 146 into such two parts, that the difference of their square roots shall be 6.
Ans. 25 and 121

7. It is required to find two numbers, such that their sum shall be 23, and their product $116\frac{1}{4}$.
Ans. $7\frac{1}{2}$ and $15\frac{1}{2}$

8. The sum of two numbers is $1\frac{1}{3}$, and the sum of their reciprocals $3\frac{1}{5}$; required the numbers. Ans. $\frac{1}{2}$ and $\frac{5}{6}$

9. The difference of two numbers is 15, and half their product is equal to the cube of the less number; required the numbers. Ans. 3 and 18

10. The difference of two numbers is 5, and the difference of their cubes 1685; required the numbers.
Ans. 8 and 13

11. A person bought a quantity of cloth for 33*l*. 15*s*. which he sold again at 2*l*. 8*s*. per piece, and gained by the bargain as much as one piece cost him; required the number of pieces. Ans. 15

12. What two numbers are those, whose sum, multiplied by the greater, is equal to 77, and whose difference, multiplied by the less, is equal to 12? Ans. 4 and 7

13. A grazier bought as many sheep as cost him 60*l.*, and after reserving 15 out of the number, sold the remainder for 54*l.*, and gained 2*s.* a head by them: how many sheep did he buy? Ans. 75

14. It is required to find two numbers, such that their product shall be equal to the difference of their squares, and the sum of their squares equal to the difference of their cubes. Ans. $\frac{1}{2}\sqrt{5}$ and $\frac{1}{4}(5+\sqrt{5})$

15. The difference of two numbers is 8, and the difference of their fourth powers is 14560; required the numbers. Ans. 3 and 11

16. A company at a tavern had 8*l.* 15*s.* to pay for their reckoning; but, before the bill was settled, two of them left the house, in consequence of which, those who remained had 10*s.* apiece more to pay than before; how many were there in company? Ans. 7

17. A person ordered 7*l.* 4*s.* to be distributed among some poor people; but, before the money was divided, there came in, unexpectedly, two claimants more, by which means the former received a shilling apiece less than they would otherwise have done; what was their number at first? Ans. 16 persons

18. It is required to find four numbers in geometrical progression such, that their sum shall be 15, and the sum of their squares 85. Ans. 1, 2, 4, and 8

19. The sum of two numbers is 11, and the sum of their fifth powers is 17831; required the numbers?

Ans. 4 and 7

20. It is required to find four numbers in arithmetical progression such, that their common difference shall be 4, and their continued product 176985.

Ans. 15, 19, 23, and 27

21. It is required to find two numbers, such that the square of the first plus their product, shall be 140, and the square of the second minus their product 78.

Ans. 7 and 13

22. Two detachments of foot being ordered to a station at the distance of 39 miles from their present quarters, began their march at the same time; but one party, by travelling $\frac{1}{4}$ of a mile an hour faster than the other, arrived there an hour sooner; required their rates of marching? Ans. $3\frac{1}{4}$ and 3 miles per hour

OF CUBIC EQUATIONS.

A cubic equation is that in which the unknown quantity rises to three dimensions; and like quadratics, or those of the higher orders, is either simple or compound.

A simple cubic equation is of the form

$$ax^3=b, \text{ or } x^3=\frac{b}{a}; \text{ where } x=\sqrt[3]{\frac{b}{a}}$$

A compound cubic equation is of the form

$$x^3+ax=b,\ x^3+ax^2=b, \text{ or } x^3+ax^2+bx=c,$$

in each of which the known quantities a, b, c, may be either $+$ or $-$.

Or, either of the two latter of these equations may be reduced to the same form as the first, by taking away its second term, which is done as follows:

RULE.

Take some new unknown quantity, and subjoin to it a third part of the coefficient of the second term of the equation with its sign changed; then if this sum, or difference, as it may happen to be, be substituted for the original unknown quantity and its powers, in the pro-

posed equation, there will arise an equation wanting its second term.

Note. The second term of any of the higher orders of equations may also be exterminated in a similar manner, by substituting for the unknown quantity the sum or difference, as above, of some other unknown quantity, and the 4th, 5th, &c., part of the coefficient of its second term, with its sign changed, according as the equation is of the 4th, 5th, &c. power.

EXAMPLES.

1. It is required to exterminate the second term of the equation $x^3+3ax^2=b$, or $x^3+3ax^2-b=0$.

$$\text{Here } x=z-\frac{3a}{3}=z-a,$$

$$\text{Then}\left\{\begin{array}{rl} x^3= & z^3-3az^2+3a^2z-a^3 \\ 3ax^2= & \quad+3az^2-6a^2z+3a^3 \\ -b= & \qquad\qquad\qquad\qquad -b \end{array}\right.$$

$$\text{Whence } z^3-3a^2z+2a^3-b=0,$$
$$\text{Or } z^3-3a^2z=b-2a^3,$$

in which equation the second power (z^2), of the unknown quantity, is wanting.

2. Let the equation $x^3-12x^2+3x=-16$, be transformed into another, that shall want the second term.

$$\text{Here } x=z+4,$$

$$\text{Then}\left\{\begin{array}{rl} (z+4)^3= & z^3+12z^2+48z+64 \\ -12(z+4)^2= & \quad-12z^2-96z-192 \\ +3(z+4)= & \qquad\qquad+3z\ +12 \end{array}\right.$$

$$\text{Whence } z^3-45z-116=-16$$
$$\text{Or } z^3-45z=100$$

which is an equation where z^2, or the second term, is wanting, as before.

3. Let the equation $x^3-6x^2=10$, be transformed into another, that shall want the second term.

Ans. $y^3-12y=26$

4. Let $y^3-15y^2+81y=243$, be transformed into an equation that shall want the second term.

Ans. $x^3+6x=88$

5. Let the equation $x^3+\frac{3}{4}x^2+\frac{7}{8}x-\frac{9}{16}=0$, be transformed into another, that shall want its second term.

Ans. $y^3+\frac{11}{16}y=\frac{3}{4}$

6. Let the equation $2x^3-3x^2+4x-5=0$, be transformed into another, that shall want its second term.

OF THE SOLUTION OF CUBIC EQUATIONS.

RULE.

Take away the second term of the equation when necessary, as directed in the preceding rule. Then, if the numeral coefficients of the given equation, or of that arising from the reduction above mentioned, be substituted for a and b in either of the following formulæ, the result will give one of the roots, as required.

$$x^3+ax=b$$

$$x=\begin{cases}\sqrt[3]{\left\{\frac{b}{2}+\sqrt{\left(\frac{b^2}{4}+\frac{a^3}{27}\right)}\right\}}+\sqrt[3]{\left\{\frac{b}{2}-\sqrt{\left(\frac{b^2}{4}+\frac{a^3}{27}\right)}\right\}} \\ \text{or} \\ \sqrt[3]{\left\{\frac{b}{2}+\sqrt{\left(\frac{b^2}{4}+\frac{a^3}{27}\right)}\right\}}-\dfrac{\frac{1}{3}a}{\sqrt[3]{\left\{\frac{b}{2}+\sqrt{\left(\frac{b^2}{4}+\frac{a^3}{27}\right)}\right\}}}\end{cases}$$

Where it is to be observed, that when the coefficient a, of the second term of the above equation, is negative, $\frac{1}{3}a$, or its cube $\frac{a^3}{27}$, in the formula, will be negative; and

if the absolute term b be negative, $\frac{b}{2}$, in the formula, will, also be negative; but $\frac{b^2}{4}$ will be positive. (e)

It may, likewise, be remarked, that when the equation is of the form

$$x^3 - ax = \pm b$$

(e) This method of solving cubic equations is usually ascribed to CARDAN, a celebrated Italian analyst of the 16th century; but the authors of it were SCIPIO FERREUS, and NICHOLAS TARTALEA, who discovered it about the same time, independently of each other, as is proved by MONTUCLA, in his *Histoire des Mathématiques*, Vol. I. p. 568, and more at large in HUTTON'S *Mathematical Dictionary*, Art. Algebra.

The rule above given, which is similar to that of Cardan, may be demonstrated as follows:

Let the equation, whose root is required, be $x^3 + ax = b$.
And assume $y + z = x$, and $3yz = -a$.

Then, by substituting these values in the given equation, we shall have $y^3 + 3y^2z + 3yz^2 + z^3 + a \times (y+z) = y^3 + z^3 + 3yz \times (y+z) + a \times (y+z) = y^3 + z^3 - a \times (y+z) + a \times (y+z) = b$; or

$$y^3 + z^3 = b.$$

And if, from the square of this last equation, there be taken 4 times the cube of the equation $yz = -\frac{1}{3}a$, we shall have $y^6 - 2y^3z^3 + z^6 = b^2 + \frac{4}{27}a^3$, or

$$y^3 - z^3 = \sqrt{(b^2 + \tfrac{4}{27}a^3)}$$

But the sum of this equation and $y^3 + z^3 = b$, is $2y^3 = b + \sqrt{(b^2 + \frac{4}{27}a^3)}$ and their difference is $2x^3 = b - \sqrt{(b^2 + \frac{4}{27}a^3)}$; whence $y = \sqrt[3]{\{\frac{1}{2}b + \sqrt{(\frac{1}{4}b^2 + \frac{1}{27}a^3)}\}}$ and $z = \sqrt[3]{\{\frac{1}{2}b - \sqrt{(\frac{1}{4}b^2 + \frac{1}{27}a^3)}\}}$

From which it appears, that $y + z$, or its equal x, is $= \sqrt[3]{\{\frac{1}{2}b + \sqrt{(\frac{1}{4}b^2 + \frac{1}{27}a^3)}\}} + \sqrt[3]{\{\frac{1}{2}b - \sqrt{(\frac{1}{4}b^2 + \frac{1}{27}a^3)}\}}$, which is the theorem above given.

Or, since z is $= -\frac{a}{3y}$, we shall have $y + z = y - \frac{a}{3y}$, or $x = \sqrt[3]{\frac{1}{2}b + \sqrt{(\frac{1}{4}b^2 + \frac{1}{27}a^3)}} - \frac{\frac{1}{3}a}{\sqrt[3]{\frac{1}{2}b + \sqrt{(\frac{1}{4}b^2 + \frac{1}{27}a^3)}}}$, being also the same as the rule.

and $\frac{a^3}{27}$ is greater than $\frac{b^2}{4}$, or $4a^3$ greater than $27b^2$, the solution of it cannot be obtained by the above rule; as the question in this instance, falls under what is usually called the *Irreducible Case* of cubic equations. (f)

EXAMPLES.

1. Given $2x^3-12x^2+36x=44$, to find the value of x.

Here $x^3-6x^2+18x=22$, by dividing by 2.

And, in order to exterminate the second term,

$$\text{Put } x=z+\frac{6}{3}=z+2,$$

$$\text{Then}\left|\begin{array}{lr}(z+2)^3=z^3+6z^2+12z+8 & \\ -6(x+2)^2= & -6z^2-24z-24 \\ 18(z+2)= & 18z+36\end{array}\right|=22$$

Whence $z^3+6z+20=22$. or $z^3+6z=2$,

And, consequently, by substituting 6 for a, and 2 for b, in the first formula, we shall have,

$$x=\sqrt[3]{\left\{\frac{2}{2}+\sqrt{\left(\frac{4}{4}+\frac{216}{27}\right)}\right\}}+\sqrt[3]{\left\{\frac{2}{2}-\sqrt{\left(\frac{4}{4}+\frac{216}{27}\right)}\right\}}=$$
$$\sqrt[3]{\{1+\sqrt{(1+8)}\}}+\sqrt[3]{\{1-\sqrt{(1+8)}\}}=\sqrt[3]{(1+\sqrt{9})}+$$
$$\sqrt[3]{(1-\sqrt{9})}=\sqrt[3]{(1+3)}+\sqrt[3]{(1-3)}=\sqrt[3]{4}-\sqrt[3]{2},$$

Therefore $x=z+2=\sqrt[3]{4}-\sqrt[3]{2}+2=2+1,587401-1.259921=2.32748$, the answer.

(f) It may here be observed, as a remarkable circumstance in the history of this science, that the solution of the *Irreducible Case* above mentioned, except by means of a table of sines, or by infinite series, has hitherto baffled the united efforts of the most celebrated mathematicians in Europe; although it is well known that all the three roots of the equation are, in this case, real; whereas, in those that are resolvable by the above formula, only one of the roots is real; so that, in fact, the rule is only applicable to such cubics as have two impossible roots.

2. Given $x^3-6x=12$, to find the value of x.

Here a being equal to -6, and b equal to 12, we shall have, by the formula,

$$x=\sqrt[3]{\{6+\sqrt{(36-8)}\}}-\frac{-2}{\sqrt[3]{\{6+\sqrt{(36-8)}\}}}=$$

$$\sqrt[3]{(6+\sqrt{28})}+\frac{2}{\sqrt[3]{(6+\sqrt{28})}}=\sqrt[3]{(6+5.2915)}+$$

$$\frac{2}{\sqrt[3]{(6+5.2915)}}=\sqrt[3]{(11.2915)}+\frac{2}{\sqrt[3]{(11.2915)}}=$$

$$2.2435+\frac{2}{2.2435}=2.2435+.8957=3.1392$$

Therefore $x=3.1392$, the answer.

3. Given $x^3-2x=-4$, to find the value of x.

Here a being $=-2$, and $b=-4$, we shall have, by the formula,

$$x=\sqrt[3]{\{-2+\sqrt{(4-\frac{8}{27})}\}}+\sqrt[3]{\{-2-\sqrt{(4-\frac{8}{27})}\}}, \text{ or}$$

by reduction, $x=\sqrt[3]{(-2+\frac{10}{9}\sqrt{3})}-\sqrt[3]{(2+\frac{10}{9}\sqrt{3})}=$

$$\sqrt[3]{(-2+1.9245)}-\sqrt[3]{(2+1.9245)}=\sqrt[3]{(-.0755)}-$$

$$\sqrt[3]{(3.9245)}=-.4226-1.5773=-1.9999, \text{ or } -2$$

Therefore $x=-2$, the answer. (g)

Note. When one of the roots of a cubic equation has been found, either by trial, or in any other way, the other two roots may be determined as follows:

Let the known root be denoted by r, and put all the

(g) When the root of the given equation is a whole number, this method determines it only by an approximation of 9s. in the decimal part, which sufficiently indicates the entire integer; but in most instances of this kind, its value may be more readily found, by a few trials, from the equation itself.

terms of the equation, when brought to the left hand side, $=0$; then if the equation, so formed, be divided by $x\mp r$, according as r is positive or negative, there will arise a quadratic equation, the roots of which will be the other two roots of the given cubic equation.

Thus, supposing $x^3-15x=4$, we can readily find, by a few trials, that $x=4$;

$$\begin{array}{l} x-4)x^3-15x-4(x^2+4x+1 \\ \quad\quad\;\; x^3-4x^2 \\ \hline \quad\quad\quad\quad 4x^2-15x \\ \quad\quad\quad\quad 4x^2-16x \\ \hline \quad\quad\quad\quad\quad\quad x-4 \\ \quad\quad\quad\quad\quad\quad x-4 \\ \hline \quad\quad\quad\quad\quad\quad\;\; * \end{array}$$

Whence, according to the note above given,

$$x^2+4x+1=0, \text{ or } x^2+4x=-1;$$

the two roots of which quadratic are $-2+\sqrt{3}$ and $-2-\sqrt{3}$; and consequently

$$4, \; -2+\sqrt{3} \text{ and } -2-\sqrt{3},$$

are the three roots of the proposed equation.

EXAMPLES FOR PRACTICE.

1. Given $x^3+3x^2-6x=8$, to find the root of the equation, or the value of x. Ans. $x=2$
2. Given $x^3+x^2=500$, to find the root of the equation, or the value of x. Ans. $x=7.617$
3. Given $x^3+12x=20$, to find the root of the equation, or the value of x. Ans. $x=1.790746$
4. Given $x^3-6x=6$, to find the root of the equation, or the value of x. Ans. $x=\sqrt[3]{2}+\sqrt[3]{4}$
5. Given $x^3+9x=6$, to find the root of the equation, or the value of x. Ans. $x=\sqrt[3]{9}-\sqrt[3]{3}$

6. Given $x^3-22x=24$, to find the root of the equation, or the value of x. Ans. $x=5.162277$

7. Given $x^3-17x^2+54x=350$, to find the root of the equation, or the value of x. Ans. $x=14.954068$

OF BIQUADRATIC EQUATIONS.

A *biquadratic equation*, as before observed, is one that rises to the fourth power, or which is of the general form

$$x^4+ax^3+bx^2+cx+d=0.$$

Or, when its second term is taken away, of the form

$$x^4+bx^2+cx+d=0,$$

to which it can always be reduced; and, in that case, its solution may be obtained by the following rule:

Find the value of z in the cubic equation

$$z^3-(\frac{1}{12}b^2+d)z=\frac{1}{108}b^3+\frac{1}{8}c^2-\frac{1}{3}bd,$$

and let the root thus determined be denoted by r.

Then find the two values of x, in each of the following quadratic equations,

$$x^2+\sqrt{\left\{2(r-\frac{1}{3}b)\right\}}x=-(r+\frac{1}{6}b)+\sqrt{\left\{(r+\frac{1}{6})b^2-d\right\}}$$

$$x^2-\sqrt{\left\{2(r-\frac{1}{3}b)\right\}}x=-(r+\frac{1}{6}b)-\sqrt{\left\{(r+\frac{1}{6}b)^2-d\right\}}$$

and they will be the four roots of the biquadratic equations required. (h)

(h) The method of solving biquadratic equations was first discovered by LOUIS FERRARI, a disciple of the celebrated

Or the four roots of the given equation, in this last case, will be as follows:

$$x = -\tfrac{1}{2}\sqrt{\left\{2(r-\frac{1}{3}b)\right\}} + \sqrt{\left\{-\frac{r}{2}+\frac{b}{3}+\sqrt{[(r+\frac{1}{6}b)^2-d]}\right\}}$$

$$x = -\tfrac{1}{2}+\left\{2(r-\frac{1}{3}b)\right\} - \sqrt{\left\{-\frac{r}{2}-\frac{b}{3}+\sqrt{[(r+\frac{1}{6}b)^2-d]}\right\}}$$

$$x = +\tfrac{1}{2}\sqrt{\left\{2(r-\frac{1}{3}b)\right\}} + \sqrt{\left\{-\frac{r}{2}-\frac{b}{3}-\sqrt{[(r+\frac{1}{6}b)^2-a]}\right\}}$$

$$x = +\tfrac{1}{2}\sqrt{\left\{2(r-\frac{1}{3}b)\right\}} - \sqrt{\left\{-\frac{r}{2}-\frac{b}{3}-\sqrt{[(r+\frac{1}{6}b)^2-d]}\right\}}$$

EXAMPLES.

1. Given $x^4+12x-17=0$, to find the four roots of the equation.

Here $a=0$, $b=0$, $c=12$, and $d=-17$;

CARDAN, before mentioned; but the above rule is derived from that given by DESCARTES in his *Geometry*, published in 1637, the truth of which may be shown as follows:

Let the equation, which is to be resolved, be

$$x^4+ax^2+bx+c=0,$$

and conceive it to be produced by the multiplication of the two quadratics

$$x^2+px+q=0, \text{ and } x^2+rx+s=0.$$

Then, since these equations, as well as the given one, are each $=0$, there will arise, by taking their product,

$$x^4+(p+r)x^3+(s+q+pr)x^2+(ps+qr)x+qs=x^4+ax^3+bx+c.$$

And, consequently, by equating the homologous terms of thi last equation, we shall have the four following equations,

$$p+r=0\ ;\ s+q+pr=a\ ;\ ps\ qr=b\ ;\ qs=c.$$

$$\text{or } r=-p,\ s+q=a+p^2,\ s-q=\frac{b}{p},\ qs=c.$$

Whence, subtracting the square of the third of these from that of the second, and then changing the sides of the equations, we shall have

Whence, by substituting these numbers in the cubic equation

$$z^3-(\frac{1}{12}b^2+d)z=\frac{1}{108}b^3+\frac{1}{8}c^2-\frac{1}{3}bd,$$

we shall have after simplifying the results,

$$z^3+17z=18,$$

Where it is evident, by inspection, that $z=1$.

And if this number be substituted for r, 0 for b, and -17 for d in the two quadratic equations in the above rule, their solution will give

$$x=-\tfrac{1}{2}\sqrt{2}+\sqrt{(-\tfrac{1}{2}+\sqrt{18})}=-\tfrac{1}{2}\sqrt{2}+\sqrt{(-\tfrac{1}{2}+3\sqrt{2})}$$
$$x=-\tfrac{1}{2}\sqrt{2}-\sqrt{(-\tfrac{1}{2}+\sqrt{18})}=-\tfrac{1}{2}\sqrt{2}-\sqrt{(-\tfrac{1}{2}+3\sqrt{2})}$$
$$x=+\tfrac{1}{2}\sqrt{2}+\sqrt{(-\tfrac{1}{2}-\sqrt{18})}=+\tfrac{1}{2}\sqrt{2}+\sqrt{(-\tfrac{1}{2}-3\sqrt{2})}$$
$$x=+\tfrac{1}{2}\sqrt{2}-\sqrt{(-\tfrac{1}{2}-\sqrt{18})}=+\tfrac{1}{2}\sqrt{2}-\sqrt{(-\tfrac{1}{2}-3\sqrt{2})}$$

Which are the four roots of the proposed equation; the two first being real, and the two last imaginary.

$$a^2+2ap^2+p^4-\frac{b^2}{p^2}=4qs, \text{ or } 4c; \text{ or } p^6+2ap^4+(a^2-4c)p^2=b^2.$$

Where, putting $p^2=z$, the value of z, and consequently of p, may be found by the rule before given for cubic equations.

Hence, also, since $s+q=a+p^2$, and $s-q=\frac{b}{p}$, there will arise, by addition and subtraction,

$$s=\frac{1}{2}a+\frac{1}{2}p^2+\frac{b}{2p};\ q=\frac{1}{2}a+\frac{1}{2}p^2-\frac{b}{2p};$$

where p being known, the values of s and q are likewise known.

And, therefore, by extracting the roots of the two assumed quadratics $x^2+px+q=0$, and $x^2+rx+s=0$, or its equal $x^2-px+s=0$, we shall have

$$x=-\tfrac{1}{2}a\pm\sqrt{(\tfrac{1}{4}p^2-q)};\ x=\tfrac{1}{2}p\pm\sqrt{(\tfrac{1}{4}p^2-s)};$$

which expressions, when taken in + and −, give the four roots of the proposed biquadratic, as was required.

Where it may be observed, that when p, in the cubic equation, $p^6+2ap^4+(a^2-4c)p^2=b^2$, is rational, the question may be solved by quadratics.

2. Given $x^4-51x^2+30x+504=0$, to find the four roots, or values of x. Ans. 3, 7, -4, and -6.

3. Given $x^4+2x^3-7x^2-8x=-12$, to find the four roots, or values, of x. Ans. 1, 2, -3, and -2.

4. Given $x^4-8x^3+14x^2+4x=8$, to find the four roots, or values of x.

$$\text{Ans.} \begin{cases} 3+\sqrt{5},\ 3-\sqrt{5} \\ 1+\sqrt{3},\ 1-\sqrt{3}. \end{cases}$$

5. Given $x^4-17x^2-20x-6=0$, to find the four roots, or values of x.

$$\text{Ans.} \begin{cases} 2+\sqrt{7}, \quad 2-\sqrt{7} \\ -2+\sqrt{2},\ -2-\sqrt{2}. \end{cases}$$

6. Given $x^4-27x^3+162x^2+356x-1200=0$, to find the four roots of the equation.

$$\text{Ans.} \begin{cases} 2.05608, \quad -3.00000 \\ 13.15306, \quad 14.79086. \end{cases}$$

7. Given $x^4-12x^2+12x-3=0$, to find the four roots of the equation.

$$\text{Ans.} \begin{cases} .606018, \quad -3.907378 \\ 2.858083, \quad .443277. \end{cases}$$

OF THE

RESOLUTION OF EQUATIONS,

BY APPROXIMATION.

EQUATIONS of the fifth power, and those of higher dimensions, cannot be resolved by any general rule, or algebraic formula, that has yet been discovered ; except in some particular cases, where certain relations subsist between the coefficients of their several terms ; or when the roots are rational, and for that reason, can be easily found by means of a few trials.

In these cases, therefore, recourse must be had to some of the usual methods of approximation ; among

which that commonly employed is the following; which is universally applicable to all kinds of numeral equations, whatever may be the number of their dimensions; and, though not strictly accurate, will give the value of the root sought to any required degree of exactness.

RULE.

Find, by trials, a number nearly equal to the root sought, which call r; and let z be made to denote the difference between this assumed root and the true root x.

Then, instead of x, in the given equation, substitute its equal $r \pm z$, and there will arise a new equation involving only z and known quantities.

Reject all the terms of this equation in which z is of two or more dimensions; and the approximate value of z may then be determined by means of a simple equation.

And if the value, thus found, be added to, or subtracted from that of r, according as r was assumed too little, or too great, it will give a near value of the root required.

But as this approximation will seldom be sufficiently exact, the operation must be repeated, by substituting the number thus found, for r, in the abridged equation exhibiting the value of z; when a second correction of z will be obtained, which, being added to, or subtracted from r, will give a nearer value of the root than the former.

And by again substituting this last number for r, in the above mentioned equation, and repeating the same process as often as may be thought necessary, a value of x may be found to any degree of accuracy required.

Note. The decimal part of the root, as found both by this and the next rule, will, in general, about double itself at each operation; and therefore it would

be useless, as well as troublesome, to use a much greater number of figures than these in the several substitutions for the values of r. (i)

EXAMPLES.

1. Given $x^3+x^2+x=90$, to find the value of x by approximation.

Here the root, as found by a few trials, is nearly equal to 4.

Let therefore $4=r$, and $r+z=x$.

$$\text{Then}\left|\begin{array}{l} x^3=r^3+3r^2z+3rz^2+z^3 \\ x^2=r^2+2rz+z^2 \\ x=r+z \end{array}\right| =90.$$

And by rejecting the terms z^3, $3rz^2$ and z^2, as small in comparison with z, we shall have

$$r^3+r^2+r+3r^2z+2rz+z=90;$$

$$\text{Whence } z=\frac{90-r^3-r^2-r}{3r^2+2r+1}=\frac{90-64-16-4}{48+8+1}=\frac{6}{57}=.10$$

And consequently $x=4.1$, *nearly.*

(i) It may here be observed, that if any of the roots of an equation be whole numbers, they may be determined by substituting 1, 2, 3, 4, *&c.*, successively, both in *plus* and in *minus*, for the unknown quantity, till a result is obtained equal to that in the question; when those that are found to succeed, will be the roots required.

Or, since the last term of any equation is always equal to the continued product of all its roots, the number of these trials may be generally diminished, by finding all the divisors of that term, and then substituting them both in *plus* and *minus*, as before, for the unknown quantity; when those that give the proper result will be the rational roots sought: but if none of them are found to succeed, it may be concluded that the equation cannot be resolved by this method; the roots, in that case, being either irrational or imaginary.

Again, if 4.1 be substituted in the place of r, in the last equation, we shall have

$$z=\frac{90-r^3-r^2-r}{3r^2+2r+1}=\frac{90-68.921-16.81-4.1}{50.43+8.2+1}=.00283$$

And consequently $x=4.1+.00283=4.10283$, for a *second approximation*.

And, if the first four figures, 4.102, of this number be again substituted for r, in the same equation, a still nearer value of the root will be obtained; and so on, as far as may be thought necessary.

2. Given $x^2+20x=100$, to find the value of x by approximation. Ans. $x=4.1421356$.

3. Given $x^3+9x^2+4x=80$, to find the value of x by approximation. Ans. $x=2.4721359$.

4. Given $x^4-38x^3+210x^2+538x+289=0$, to find the value of x by approximation.

Ans. $x=30.53565375$.

5. Given $x^5+6x^4-10x^3-112x^2-207x+110=0$, to find the value of x by approximation.

Ans. 4.46410161.

The roots of equations, of all orders, can also be determined, to any degree of exactness, by means of the following easy rule of double position; which, though it has not been generally employed for this purpose, will be found, in some respects, superior to the former, as it can be applied, at once, to any unreduced equation, consisting of surds, or compound quantities, as readily as if it had been brought to its usual form*.

RULE.

Find, by trial, two numbers nearly equal to the root sought, and substitute them in the given equation instead of the unknown quantity, noting the results that are obtained from each.

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* NOTE. Another method of approximating to the roots of equations will be found in the *Addenda*. ED.

Ceaf

Then, as the difference of these results is to the difference of the two assumed numbers, so is the difference between the true result, given by the question, and either of the former, to the correction of the number belonging to the result used; which correction being added to that number when it is too little, or subtracted from it when it is too great, will give the root required, *nearly*.

And if the number thus determined, and the nearest of the two former, or any other that appears to be more accurate, be now taken as the assumed roots, and the operation be repeated as before, a new value of the unknown quantity will be obtained still more correct than the first; and so on, proceeding in this manner, as far as may be judged necessary. (k)

(k) The above rule of Double Position, which is much more simple and commodious than the one commonly employed for this purpose, is the same as that which was first given at p. 311 of the octavo edition of my *Arithmetic*, published in 1810.

To this we may farther add, that when one of the roots of an equation has been found, either by this method or the former, the other roots may be determined as follows:

Bring all the terms to the left hand side of the equation, and divide the whole expression, so formed, by the difference between the unknown quantity (x) and the root first found; and the resulting equation will then be depressed a degree lower than the given one.

Find a root of this new equation, by approximation, as in the first instance, and the number so obtained will be a second root of the original equation.

Then, by means of this root, and the unknown quantity, depress this second equation a degree lower, and thence find a third root; and so on, till the equation is reduced to a quadratic; when the two roots of this, together with the former, will be the roots of the equation required.

Thus, in the equation $x^3 - 15x^2 + 63x = 50$, the first root is found, by approximation, to be 1.02804. Hence

$x - 1.02804)x^3 - 15x^2 + 63x - 50(x^2 - 13.97196x + 48.63627 = 0.$

And the two roots of the quadratic equation, $x^2 - 13.97196x = -48.63627$, found in the usual way, are 6.57653 and 7.39543.

So

EXAMPLES.

1. Given $x^3+x^2+x=100$, to find an approximate value of x.

Here it is soon found, by a few trials, that the value of x lies between 4 and 5.

Hence, by taking these as the two assumed numbers, the operation will stand as follows:

	First Sup.		Second Sup.	
	4	x	5	
	16	x^2	25	
	64	x^3	125	
	84	Results	155	
Therefore	155	5	100	
	84	4	84	
	71 :	1 ::	16 :	225

And consequently $x=4+.225=4.225$, *nearly*.

Again, if 4.2 and 4.3 be now taken as the two assumed numbers, the operation will stand thus:

	First Sup.		Second Sup.	
	4.2	x	4.3	
	17.64	x^2	18.49	
	74.088	x^3	79.507	
	95.928	Results	102.297	
	102.297	4.3	102.297	
	95.928	4.2	100	
Therefore	6.369 :	.1 ::	2.297 :	.036.

So that the three roots of the given cubic equation, $x^3-15x^2+63x=50$ are 1.02804, 6.57653, and 7.39543; their sum being $=15$, the coefficient of the second term of the equation, as it ought to be when they are right.

And consequently $x = 4.3 - .036 = 4.264$, *nearly.*

Again, let 4.264 and 4.265 be the two assumed numbers; then

First Sup.		*Second Sup.*
4.264	. . x . .	4.265
18.181696	. . x^2 . .	18.190225
77.526752	. . x^3 . .	77.581310
99.972448	Results	100.036535

Therefore

100.036535	4.265	100
99.972448	4.264	99.972448
.064087 :	.001 ::	.027552 : .0004299.

And consequently

$x = 4.264 + .0004299 = 4.2644299$, *very nearly.*

2. Given $(\frac{1}{5}x^2 - 15)^2 + x\sqrt{x} = 90$, to find an approximate value of x.

Here, by a few trials, it will be soon found, that the value of x lies between 10 and 11; which let, therefore, be the two assumed numbers, agreeably to the directions given in the rule.

Then

First Sup.		*Second Sup.*
25	. . $(\frac{1}{5}x^2 - 15)^2$.	.84.64
31.622	. . $x\sqrt{x}$.	.36.482
56.622	Results	121.122

Hence

121.122 . .	11 . .	121.122
56.622 . .	10 . .	90
64.5 :	1 ::	31.122 : .482,

And consequently $x = 11 - .482 = 10.518$,

Again, let 10.5 and 10.6 be the two assumed numbers,

Then

First Sup.		Second Sup.
49.7025 . .	$(\frac{1}{5}x^2-15)^2$. .	55.830784
34.0239 . .	$x\sqrt{x}$. .	34.511099
83.7264 . .	Results . .	90.341883

Hence

90.341883. .	10.6 . .	90.341883
83.7264 . .	10.5 . .	90.
6.615483 :	.1 ::	.341883 : .0051679.

And consequently

$x=10.6-.0051679=10.5948321$, *very nearly*.

EXAMPLES FOR PRACTICE.

1. Given $x^3+10x^2+5x=2600$, to find a near approximate value of x. Ans. $x=11.00673$
2. Given $2x^4-16x^3+40x^2-30x+1=0$, to find a near value of x. Ans. $x=1.284724$
3. Given $x^5+2x^4+3x^3+4x^2+5x=54321$, to find the value of x. Ans. 8.414455
4. Given $\sqrt[3]{(7x^3+4x^2)}+\sqrt{(20x^2-10x)}=28$, to find the value of x. Ans. 4.510661
5. Given $\sqrt{\{144x^2-(x^2+20)^2\}}+\sqrt{\{196x^2-(x^2+24)^2\}}=114$, to find the value of x. Ans. 7.123883

Of EXPONENTIAL EQUATIONS.

An exponential quantity is that which is to be raised to some unknown power, or which has a variable quantity for its index, as

$$a^x,\ a^{\frac{1}{x}},\ x^x, \text{ or } x^{\frac{1}{x}}, \text{ \&c.}$$

And an exponential equation is that which is formed between any expression of this kind and some other quantity whose value is known: as

$$a^x=b,\ x^x=a, \text{ \&c.}$$

where it is to be observed, that the first of these equations, when converted into logarithms, is the same as

$$x \log. a = \log. b, \text{ or } x = \frac{\log. b,}{\log. a.}$$

And the second, $x^x = a$, is the same as

$$x \log. x = \log. a.$$

In the latter of which cases, a near approximate value of the unknown quantity may be determined, as follows:

RULE.

Find, by trial, two numbers as near as can conveniently be done, to the number sought, and substitute them in the given equation.

$$x \log. x = \log. a,$$

instead of the unknown quantity, noting the results obtained from each, as in the rule of Double Position before laid down.

Then, by means of a certain number of successive operations, performed in the same manner as is there described, the value of x may be found to any degree of accuracy required. (l) *

EXAMPLES.

1. Given $x^x = 100$, to find an approximate value of x.

Here, by the above formula, we have

$$x \log. x = \log. 100 = 2.$$

And since x is readily found, by a few trials, to be

(l) Many attempts have been made to determine the value of the unknown quantity, in the exponential equation $x^x = a$, above given, by converting it into a series, the terms of which shall consist only of a and its powers; but no expression of this kind has hitherto been discovered, which is sufficiently convergent to answer any practical purpose. See Vol. II. of my *Treatise on Algebra*, before referred to.

* NOTE. Another method of approximating to the roots of exponential equations will be found in the *Addenda*. ED.

nearly in the middle between 3 and 4, but rather nearer the latter than the former, let 3.5 and 3.6 be taken for the two assumed numbers.

Then log. $3.5 = .5440680$; which, being multiplied by 3.5, gives $1.904238 =$ first result:

And log. $3.6 = .5563025$; which, being multiplied by 3.6, gives 2.002689 for the second result.

Whence

$$\begin{array}{ccccc} 2.002689 & . . & 3.6 & . . & 2.002689 \\ .904238 & . . & 3.5 & . . & 2. \\ \hline 1.098451 & : & .1 & :: & .002689 : 00273 \end{array}$$

For the first correction; which, taken from 3.6, leaves $x = 3.59727$, *nearly*.

And this value is found, by trial, to be rather too small, let 3.59727 and 3.59728 be taken as the two assumed numbers.

Then log. $3.59727 = .5559731$; which, being multiplied by 3.59727, gives $1.9999854 =$ first result.

And log. $3.59728 = .5559743$; which, being multiplied by 3.51728, gives $1.9999953 =$ second result.

Whence

$$\begin{array}{ccccc} 1.9999953 & . . & 3.59728 & . . & 2. \\ 1.9999854 & . . & 3.59727 & . . & 1.9999953 \\ \hline .0000099 & : & .00001 & :: & .0000047 : .0000047474 7 \end{array}$$

for the second correction; which, added to 3.59728, gives $x = 3.59728474747$, the answer required; being a value of x extremely near the truth.

2. Given $x^x = 2000$, to find an approximate value of x.
Ans. $x = 4.82782263$

3. Given $(6x)^x = 96$, to find an approximate value of x.
Ans. $x = 1.8826432$

4. Given $x^x = 123456789$, to find an approximate value of x.
Ans. 8.6400268

5. Given $x^x - x = (2x - x^x)^{\frac{1}{x}}$, to find an approximate value of x. Ans. $x = 1.747933$

OF THE

BINOMIAL THEOREM.

THE binomial theorem is a general algebraical expression or formula, by which any power, or root of a given quantity, consisting of two terms, is expanded into a series, the form of which, as it was first proposed by Newton, being as follows:

$$(P+PQ)^{\frac{m}{n}} = P^{\frac{m}{n}}[1 + \frac{m}{n}Q + \frac{m}{n}(\frac{m-n}{2n})Q^2 + \frac{m}{n}(\frac{m-n+}{2n})$$

$$(\frac{m-2n}{3n})Q^3 + \frac{m}{n}(\frac{m-n}{2n})(\frac{m-2n}{3n})(\frac{m-3n}{4n}) \text{ \&c.}$$

$$(P+PQ)^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ +$$

$$\frac{m-3n}{4n}DQ + \frac{m-4n}{5n}EQ \text{ \&c.}$$

Where P is the first term of the binomial, Q the second term divided by the first, $\frac{m}{n}$ the index of the power, or root, and A, B, C, &c., the terms immediately preceding those in which they are first found, including their signs + or —.

Which theorem may be readily applied to any particular case, by substituting the numbers, or letters, in the given example, for P, Q, m, and n, in either of the above formulæ, and then finding the result according to the rule. (l)

(l) This celebrated theorem, which is of the most extensive

EXAMPLES.

1. It is required to convert $(a^2+x)^{\frac{1}{2}}$ into an infinite series.

Here $P=a^2$, $Q=\frac{x}{a^2}$, $\frac{m}{n}=\frac{1}{2}$, or $m=1$, and $n=2$.

Whence

$$P^{\frac{m}{n}}=(a^2)^{\frac{m}{n}}=(a^2)^{\frac{1}{2}}=a=A,$$

$$\frac{m}{n}AQ=\frac{1}{2}\times\frac{a}{1}\times\frac{x}{a^2}=\frac{x}{2a}=B.$$

---

use in algebra, and various other branches of analysis, may be otherwise expressed as follows:

$$(a+x)^{\frac{m}{n}}=a^{\frac{m}{n}}[1+\frac{m}{n}(\frac{x}{a})+\frac{m}{n}.\frac{m-n}{2n}(\frac{x}{a})^2+\frac{m}{n}.\frac{m-n}{2n}.\frac{m-2n}{3n}$$

$$\times(\frac{x}{a})^3+\&c.]$$

$$\text{Or } (a+x)^{\frac{m}{n}}=$$

$$a^{\frac{m}{n}}[1+\frac{m}{n}(\frac{x}{a+x})+\frac{m}{n}.\frac{m+n}{2n}(\frac{x}{a+x})^2+\frac{m}{n}.\frac{m+n}{3n}.\frac{m+2n}{3n}$$

$$\times(\frac{x}{a+x})^3+\&c.]$$

$$\text{Or } (a+x)^{\frac{m}{n}}=$$

$$2a^{\frac{m}{n}}[1-\frac{m}{n}(\frac{a-x}{a+x})+\frac{m}{n}.\frac{m+n}{2n}(\frac{a-x}{a+x})^2-\frac{m}{n}.\frac{m+n}{2n}.\frac{n+2n}{3n}$$

$$\times(\frac{a-x}{a+x})^3+\&c.]$$

It may here also be observed, that if m be made to represent any whole, or fractional number, whether positive or

$$\frac{m-n}{2n}\,\text{BQ}=\frac{1-2}{4}\times\frac{x}{2a}\times\frac{x}{a^2}=-\frac{x^2}{2.4a^3}=\text{C},$$

$$\frac{m-2n}{3n}\,\text{CQ}=\frac{1-4}{6}\times-\frac{x^2}{2.4a^3}\times\frac{a}{a^2}=\frac{3x^3}{2.4.6a^5}=\text{D},$$

$$\frac{m-3n}{4n}\,\text{DQ}=\frac{1-6}{8}\times\frac{3x^3}{2.4.6a^5}\times\frac{x}{a^2}=-\frac{3.5x^4}{2.4.6.8a^7}=\text{E},$$

$$\frac{m-4n}{5n}\,\text{EQ}=\frac{1-8}{10}\times-\frac{3.5x^4}{2.4.6.8a^7}\times\frac{x}{a^2}=\frac{3.5.7x^5}{2.4.6.8.10a^9}=\text{F},$$

&c. &c. &c.

Therefore $(a^2+x)^{\frac{1}{2}}=$

$$a+\frac{x}{2a}-\frac{x^2}{2.4a^3}+\frac{3x^3}{2.4.6a^5}-\frac{3.5x^4}{2.4.6.8a^7}+\frac{3.5.7x^5}{2.4.6.8.10a^9}-\text{\&c.}$$

Where the law of formation of the several terms of the series is sufficiently evident.

2. It is required to convert $\frac{1}{(a+b)^2}$ or its equal $(a+b)^{-2}$, into an infinite series.

Here $\text{P}=a$, $\text{Q}=\frac{b}{a}$, and $\frac{m}{n}=-2$, or $m=-2$ and $n=1$; whence

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negative, the first of these expressions may be exhibited in the more simple form

$$(a+x)^m=a^m+ma^{m-1}x+\frac{m(m-1)}{1\,.\,2}a^{m-2}x^2+\frac{m(m-1)}{1\,.\,2}\,\frac{(m-2)}{3}a^{m-3}x^3\ldots$$

$$\ldots\ldots\frac{m(m-1)\,(m-2)\,\ldots\ldots\,[m-(n-1)]a^nx^{m-n}}{1\,.\,2\,.\,3\,.\,4\,\ldots\ldots\ldots\,n}$$

Where the last term is called the *general term of the series*, because if 1, 2, 3, 4, *&c.*, be substituted successively for n, it will give all the rest.

$$P^{\frac{m}{n}}=(a)^{\frac{m}{n}}=a^{-2}=\frac{1}{a^2}=A,$$

$$\frac{m}{n}AQ=-\frac{2}{1}\times\frac{1}{a^2}\times\frac{b}{a}=-\frac{2b}{a^3}=B,$$

$$\frac{m-n}{2n}BQ=\frac{-2-1}{2}\times-\frac{2b}{a^3}\times\frac{b}{a}=\frac{3b^2}{a^4}=C,$$

$$\frac{m-2n}{3n}CQ=\frac{-2-2}{3}\times\frac{3b^2}{a^4}\times\frac{b}{a}=-\frac{4b^3}{a^5}=D,$$

$$\frac{m-3n}{4n}DQ=\frac{-2-3}{4}\times-\frac{4b^3}{a^5}\times\frac{b}{a}=\frac{5b^4}{a^6}=E,$$

&c. &c. &c.

Consequently $\frac{1}{(a+b)^2}=\frac{1}{a^2}-\frac{2b}{a^3}+\frac{3b^2}{a^4}-\frac{4b^3}{a^5}+\frac{5b^4}{a^6}$ &c.

3. It is required to convert $\frac{a^2}{(a^2-x)^{\frac{1}{2}}}$, or its equal $a^2(a^2-x)^{-\frac{1}{2}}$, into an infinite series.

Here

$P=a^2$, $Q=-\frac{x}{a^2}$, and $\frac{m}{n}=\frac{-1}{2}$, or $m=-1$ and $n=2$ whence

$$P^{\frac{m}{n}}=(a^2)^{\frac{m}{n}}=(a^2)^{-\frac{1}{2}}=\frac{1}{a}=A,$$

$$\frac{m}{n}AQ=-\frac{1}{2}\times\frac{1}{a}\times-\frac{x}{a^2}=\frac{x}{2a^3}=B,$$

$$\frac{m-n}{2n}BQ=\frac{-1-2}{4}\times\frac{x}{2a^3}\times-\frac{x}{a^2}=\frac{3x^2}{2.4a^5}=C,$$

$$\frac{m-2n}{3n}CQ=\frac{-1-4}{6}\times\frac{3x^2}{2.4a^5}\times-\frac{x}{a^2}=\frac{3.5.x^3}{2.4.6a^7}=D,$$

$$\frac{m-3n}{4n}\text{DQ}=\frac{-1-6}{8}\times\frac{3.5x^3}{2.4.6a^7}\times-\frac{x}{a^2}=\frac{3.5.7x^4}{2.4.6.8a^9}=\text{E},$$

&c. &c. &c.

Therefore

$$\frac{1}{(a^2-x)^{\frac{1}{2}}}=\frac{1}{a}+\frac{1}{2}(\frac{x}{a^3})+\frac{3}{2.4}(\frac{x^2}{a^5})+\frac{3.5}{2.4.6}(\frac{x^3}{a^5})+\frac{3.5.7}{2.4.6.8}(\frac{x^4}{a^9})$$
$+$ &c.

And

$$\frac{a^2}{(a^2-x)^{\frac{1}{2}}}=a+\frac{1}{2}(\frac{x}{a})+\frac{3}{2.4}(\frac{x^2}{a^3})+\frac{3.5}{2.4.6}(\frac{x^3}{a^5})+\frac{3.5.7}{2.4.6.8}(\frac{x^4}{a^8})$$
$+$ &c.

4. It is required to convert $\sqrt[3]{9}$, or its equal $(8+1)^{\frac{1}{3}}$ into an infinite series.

Here $\text{P}=8$, $\text{Q}=\frac{1}{8}$, and $\frac{m}{n}=\frac{1}{3}$, or $m=1$ and $n=3$;

Whence

$$\text{P}^{\frac{m}{n}}=(8)^{\frac{m}{n}}=8^{\frac{1}{3}}=2=\text{A},$$

$$\frac{m}{n}\text{AQ}=\frac{1}{3}\times\frac{2}{1}\times\frac{1}{2^3}=\frac{1}{3.2^2}=\text{B},$$

$$\frac{m-n}{2n}\text{BQ}=\frac{1-3}{6}\times\frac{1}{3.2^2}\times\frac{1}{2^3}=-\frac{1}{3.6.2^4}=\text{C},$$

$$\frac{m-2n}{3n}\text{CQ}=\frac{1-6}{9}\times-\frac{1}{3.6.2^4}\times\frac{1}{2^3}=\frac{5}{3.6.9.2^7}=\text{D},$$

$$\frac{m-3n}{4n}\text{DQ}=\frac{1-9}{12}\times\frac{5}{3.6.9.2^7}\times\frac{1}{2^3}=-\frac{5.8}{3.6.9.12.2^{10}}=\text{E},$$

$$\frac{m-4n}{5n}\text{EQ}=\frac{1-12}{15}\times-\frac{5.8}{3.6.9.12.2^{10}}\times\frac{1}{2^3}=\frac{5.8.11}{3.6.9.12.15.2^{13}}$$

&c. &c. &c.

Therefore $\sqrt[3]{9}=$

$$2+\frac{1}{3.2^2}-\frac{1}{3.6.2^4}+\frac{5}{3.6.9.2^7}-\frac{5.8}{3.6.9.12.2^{10}}+\frac{5.8.11}{3.6.9.12.15.2^{13}}$$
$-$ &c.

5. It is required to convert $\sqrt{2}$, or its equal $\sqrt{(1+1)}$, into an infinite series. $=(1+1)^{\frac{1}{2}}$

Ans. $1+\frac{1}{2}-\frac{1}{2.4}+\frac{3}{2.4.6}-\frac{3.5}{2.4.6.8}+\frac{3.5.7}{2.4.6.8.10}-$ &c.

6. It is required to convert $\sqrt[3]{7}$, or its equal $(8-1)^{\frac{1}{3}}$, into an infinite series.

Ans. $2-\frac{1}{3.2^2}-\frac{1}{3.6.2^4}-\frac{5}{3.6.9.2^7}-\frac{5.8}{3.6.9.12.2^{10}}-$ &c.

7. It is required to convert $\sqrt[5]{240}$, or its equal $(243-3)^{\frac{1}{5}}$, into an infinite series.

Ans. $3-\frac{1}{5.3^3}-\frac{4}{5.10.3^7}-\frac{4.9}{5.10.15.3^{11}}-\frac{4.9.14}{5.10.15.20.3^{15}}$
$-$ &c.

8. It is required to convert $(a\pm x)^{\frac{1}{2}}$ into an infinite series.

Ans. $a^{\frac{1}{2}}\{1\pm\frac{x}{2a}-\frac{x^2}{2.4a^2}\pm\frac{3.x^3}{2.4.6a^3}-\frac{3.5x^4}{2.4.6.8a^4}\pm$ &c.$\}$

9. It is required to convert $(a\pm b)^{\frac{1}{3}}$ into an infinite series.

Ans. $a^{\frac{1}{3}}\{1\pm\frac{b}{3a}-\frac{2b^2}{3.6a^2}\pm\frac{2.5b^3}{3.6.9a^3}-\frac{2.5.8b^4}{3.6.9.12a^4}\pm$ &c.$\}$

10. It is required to convert $(a-b)^{\frac{1}{4}}$ into an infinite series.

Ans. $a^{\frac{1}{4}}\{1-\frac{b}{4a}-\frac{3b^2}{4.8a^2}-\frac{3.7b^3}{4.8.12a^3}-\frac{3.7.11b^4}{4.8.12.16a^4}-$ &c.$\}$

11. It is required to convert $(a+x)^{\frac{2}{3}}$ into an infinite series.

$$\text{Ans. } a^{\frac{2}{3}}\left\{1+\frac{2x}{3a}-\frac{x^2}{9a^2}+\frac{4x^3}{9^2a^3}-\frac{4.7x^4}{9^212a^4}+\frac{4.7.10x^5}{9.^212.15a^5}-\&c.\right\}$$

12. It is required to convert $(1-x)^{\frac{2}{5}}$ into an infinite series.

$$\text{Ans. } 1-\frac{2x}{5}-\frac{2.3x^2}{5.10}-\frac{2.3.8x^3}{5.10.15}-\frac{2.3.8.13x^4}{5.10.15.20}-\&c.$$

13. It is required to convert $\frac{1}{(a\pm x)^{\frac{1}{2}}}$, or its equal $(a+x)^{-\frac{1}{2}}$, into an infinite series.

$$\text{Ans. } \frac{1}{a^{\frac{1}{2}}}\left\{1\mp\frac{x}{3a}+\frac{3x^2}{2.4a^2}\mp\frac{3.5x^3}{2.4.6a^3}+\frac{3.5.7x^4}{2.4.6.8a^4}\mp\&c.\right\}$$

14. It is required to convert $\frac{a}{(a\pm x)^{\frac{1}{3}}}$, or its equal $a(a+x)^{-\frac{1}{3}}$, into an infinite series.

$$\text{Ans. } a^{\frac{2}{3}}\left\{1\mp\frac{x}{3a}+\frac{4x^2}{3.6.a^2}\mp\frac{4.7x^3}{3.6.9a^3}+\frac{4.7.10x^4}{3.6.9.12a^4}\mp\&c.\right\}$$

15. It is required to convert $\frac{1}{(1+x)^{\frac{1}{5}}}$ or its equal $(1+x)^{-\frac{1}{5}}$, into an infinite series.

$$\text{Ans. } 1-\frac{x}{5}+\frac{6x^2}{5.10}-\frac{6.11x^3}{5.10.15}+\frac{6.11.16x^4}{5.10.15.20}-\&c.$$

16. It is required to convert $\left(\frac{a+x}{a-x}\right)^{\frac{1}{2}}$, or its equal $(a+x)(a^2-x^2)^{-\frac{1}{2}}$, into an infinite series.

$$\text{Ans. } 1+\frac{x}{a}+\frac{x^2}{2a^2}+\frac{x^3}{2a^3}+\frac{3x^4}{8a^4}+\frac{3x^5}{8a^5}+\frac{5x^6}{16x^6}+\frac{5x^7}{16a^7}+\&c.$$

OF THE

INDETERMINATE ANALYSIS.

In the common rules of algebra, such questions are usually proposed as require some certain or definite answer; in which case, it is necessary that there should be as many independent equations, expressing their conditions, as there are unknown quantities to be determined; or otherwise the problem would not be limited.

But in other branches of the science, questions frequently arise that involve a greater number of unknown quantities than there are equations to express them; in which instances they are called indeterminate or unlimited problems; being such as usually admit of an indefinite number of solutions; although, when the question is proposed in integers, and the answers are required only in whole positive numbers, they are, in some cases, confined within certain limits, and in others, the problem may become impossible.

PROBLEM I.

To find the integral values of the unknown quantities x and y in the equation.

$$ax-by=\pm c, \text{ or } ax+by=c.$$

Where a and b are supposed to be given whole numbers, which admit of no common divisor, except when it is also a divisor of c.

RULE.

1. Let wh denote a whole or integral number; and reduce the equation to the form

$$x=\frac{by\pm c}{a}wh, \text{ or } x=\frac{c-by}{a}wh.$$

2. Throw all whole numbers out of that of these two

expressions, to which the question belongs, so that the numbers d and e in the remaining parts, may be each less than a; then

$$\frac{dy \pm e}{a} = wh, \text{ or } \frac{e - dy}{a} = wh.$$

3. Take such a multiple of one of these last formulæ, corresponding with that above mentioned, as will make the coefficient of y nearly equal to a, and throw the whole numbers out of it as before.

Or find the sum or difference of $\frac{ay}{a}$ and the expression above used, or any multiple of it that comes near $\frac{ay}{a}$, and the result, in either of these cases, will still be a whole number.

4. Proceed in the same manner with this last result; and so on, till the coefficient of y becomes equal to 1, and the remainder equal to some number r; then

$$\frac{y \pm r}{a} = wh. = p, \text{ and } y = ap \mp r,$$

Where p may be o, or any integral number whatever, that makes y positive.

And as the value of y is now known, that of x may be found from the given equation, when the question is possible. (m)

NOTE. Any indeterminate equation of the form

$$ax - by = \pm c,$$

in which a and b are prime to each other, is always possible, and will admit of an infinite number of answers in whole numbers.

(m) This rule is founded on the obvious principle, that the sum, difference, or product of any two whole numbers, is a whole number; and that, if a number divides the whole of any other number and a part of it, it will also divide the remaining part.

But if the proposed equation be of the form

$$ax+by=c,$$

the number of answers will always be limited; and in some cases the question is impossible; both of which circumstances may be readily discovered, from the mode of solution above given. (n)

EXAMPLES.

1. Given $19x-14y=11$, to find x and y in whole numbers.

Here $x=\dfrac{14y+11}{19}=wh.$, and also $\dfrac{19y}{19}=wh.$

Whence, by subtraction, $\dfrac{19y}{19}-\dfrac{14y+11}{19}=\dfrac{5y-11}{19}=wh.$

Also, $\dfrac{5y-11}{19}\times 4=\dfrac{20y-44}{19}=y-2+\dfrac{y-6}{19}=wh.$

And by rejecting $y-2$, which is a whole number,

$$\frac{y-6}{19}=wh.=p.$$

Whence we have $y=19p+6$,

(n) That the coefficients a and b, when these two formulæ are possible, should have no common divisor, which is not, at the same time, a divisor of c, is evident; for if $a=md$, and $b=me$, we shall have $ax\pm by=mdx\pm mey=c$; and consequently

$$dx+ey=\frac{c}{m}$$

But d, e, x and y, being supposed to be whole numbers, $\dfrac{c}{m}$ must also be a whole number, which it cannot be, except when m is a divisor of c.

Hence, if it were required to pay 100*l.* in guineas and moidores only, the question would be impossible; since, in the equation $21x+27y=2000$, which represents the conditions of the problem, the coefficients, 21 and 27, are each divisible by 3, whilst the absolute term 2000 is not divisible by it. See Vol. II. of my *Treatise on Algebra*, for the method of resolving questions of this kind, by means of *Continued Fractions.*

And $x=\frac{14y+11}{19}=\frac{14(19p+6)+11}{19}=\frac{266p+95}{19}=$
$$14p+5.$$

Where if p be taken $=0$ we shall have $x=5$ and $y=6$, for their least values; the number of solutions being obviously indefinite.

2. Given $2x+3y=25$, to determine x and y in whole positive numbers.

$$\text{Here } x=\frac{25-3y}{2}=12-y+\frac{1-y}{2}$$

Hence, since x must be a whole number, it follows that $\frac{1-y}{2}$ must also be a whole number.

$$\text{Let therefore } \frac{1-y}{2}=wh=p;$$

$$\text{Then } 1-y=2p, \text{ or } y=1-2p.$$

And since

$$x=12-y+\frac{1-y}{2}=12-(1-2p)+p=12+3p-1,$$

We shall have $x=11+3p$, and $y=1-2p$; Where p may be any whole number whatever, that will render the values of x and y in these two equations positive.

But it is evident, from the value of y, that p must be either 0 or negative; and, consequently, from that of x, that it must be 0, -1, -2, or -3.

Whence, if $p=0$, $p=-1$, $p=-2$, $p=-3$,

$$\text{Then } \begin{cases} x=11, & x=8, & x=5, & x=2 \\ y=1, & y=3, & y=5, & y=7 \end{cases}$$

Which are all the answers in whole positive numbers that the question admits of.

3. Given $3x=8y-16$, to find the values of x and y in whole numbers.

Here $x=\frac{8y-16}{3}=2y-5+\frac{2y-1}{3}=wh.$; or $\frac{2y-1}{3}=wh$

$$\text{Also } \frac{2y-1}{3}\times 2=\frac{4y-2}{3}=y+\frac{y-2}{3}=wh.$$

Or, by rejecting y, which is a whole number, there will remain $\frac{y-2}{3}=wh.=p.$

$=\frac{8(3p+2)-16}{3}=$

$$\text{Therefore } y=3p+2,$$

$$\text{And } x=\frac{8y-16}{3}=\frac{8(3p+2)-16}{3}=\frac{24p}{3}=8p.$$

Where if p be put $=1$, we shall have $x=8$ and $y=5$, for their least values; the number of answers being, as in the first question indefinite.

4. Given $21x+17y=2000$, to find all the possible values of x and y in whole numbers.

$$\text{Here } x=\frac{2000-17y}{21}=95+\frac{5-17y}{21}=wh.;$$

$$\text{Or, omitting the 95, } \frac{5-17y}{21}=wh.;$$

Consequently, by addition, $\frac{21y}{21}+\frac{5-17y}{21}=\frac{4y+5}{21}=wh.$

$$\text{Also, } \frac{4y+5}{21}\times 5=\frac{20y+25}{21}=1+\frac{4+20y}{21}=wh.;$$

Or, by rejecting the whole number, 1, $\frac{4+20y}{21}=wh.$

And, by subtraction, $\frac{21y}{21}-\frac{4+20y}{21}=\frac{y-4}{21}=wh,=p;$

$$\text{Whence } y=21p+4,$$

$$\text{And } x=\frac{2000-17y}{21}=\frac{2000-17(21p+4)}{21}=92-17p.$$

Where, if p be put $=0$, we shall have the least value of $y=4$, and the corresponding, or greatest value of $x=92$.

And the rest of the answers will be found by adding 21 continually to the least value of y, and subtracting 17 from the greatest value of x; which being done we shall obtain the six following results:

$x=92$	75	58	41	24	7
$y=4$	25	46	67	88	109

These being all the solutions, in whole numbers, that the question admits of.

Note 1. When there are three or more unknown quantities, and only one equation by which they can be determined, as

$$ax+by+cz=d,$$

it will be proper first to find the limit of the quantity that has the greatest coefficient, and then to ascertain the different values of the rest, by separate substitutions of the several values of the former, from 1 up to that extent, as in the following question.

5. Given $3x+5y+7z=100$, to find all the different values of x, y, and z, in whole numbers. (*o*)

Here each of the least integer values of x and y are 1, by the question; whence it follows, that

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(*o*) If any indeterminate equation, of the kind above given, has one or more of its coefficients, as c negative, the equation may be put under the form

$$ax+by=d+cz,$$

in which case it is evident that an indefinite number of values may be given to the second side of the equation, by means of the indefinite quantity z; and consequently, also, to x and y, in the first.

And if the coefficients a, b, c, in any such equation, have a common divisor, while the absolute number d has not, the question, as in the first case, becomes impossible. For the reason of which, see Vol. I. of my *Treatise on Algebra*, before quoted.

$$z=\frac{100-5-3}{7}=\frac{100-8}{7}=\frac{92}{7}=13\tfrac{1}{7}.$$

Consequently z cannot be greater than 13, which is also the limit of the number of answers; though they may be considerably less.

By proceeding, therefore, as in the former rule, we shall have

$$x=\frac{100-5y-7z}{3}=33-y-2z+\frac{1-2y-z}{3}=wh.;$$

And, by rejecting $33-y-2z$,

$$\frac{1-2y-z}{3}=wh.; \text{ or } \frac{3y}{3}+\frac{1-2y-z}{3}=\frac{y+1-z}{3}=wh.$$

$$\text{Whence } \frac{y+1-z}{3}=p.$$

$$\text{Or } y=3p+z-1;$$

And consequently, putting $p=0$, we shall have the least value of $y=z-1$; where z may be any number, from 1 up to 13, that will answer the conditions of the question.

When, therefore, $z=2$, we have $y=1$,

$$\text{And } x=\frac{100-19}{3}=27.$$

Hence, by taking $z=2, 3, 4, 5$, &c., the corresponding values of x and y, together with those of z, will be found to be as below

$z=2$	3	4	5	6	7	8
$y=1$	2	3	4	5	6	7
$x=27$	23	19	15	11	7	3

Which are all the integral values of x, y, and z, that can be obtained from the given equation.

Note 2. If there be three unknown quantities, and only two equations for determining them, as

$$ax+by+cz=d, \text{ and } ex+fy+gz=h,$$

exterminate one of these quantities in the usual way, and find the values of the other two from the resulting equation, as before.

Then, if the values, thus found, be separately substituted in either of the given equations, the corresponding values of the remaining quantities will likewise, be determined: Thus,

6. Let there be given $x-2y+z=5$, and $2x+y-z=7$, to find the values of x, y, and z.

Here, by multiplying the first of these equations by 2, and subtracting the second from the product, we shall have

$$3z-5y=3, \text{ or } z=\frac{3+5y}{3}=1+y+\frac{2y}{3}=wh.;$$

And consequently $\frac{2y}{3}$, or $\frac{3y}{3}-\frac{2y}{3}=\frac{y}{3}=wh.=p$,

Whence $y=3p$.

And, by taking $p=1, 2, 3, 4$, &c., we shall have $y=3, 6, 9, 12, 15$, &c., and $z=6, 11, 16, 21, 26$, &c.

But from the first of the two given equations.

$$x=5+2y-z;$$

Whence, by substituting the above values for y and z, the results will give $x=5, 6, 7, 8, 9$, &c.

And therefore the first six values of x, y, and z, are as below:

$x=5$	6	7	8	9	10
$y=3$	6	9	12	15	18
$z=6$	11	16	21	26	31

Where the law by which they can be continued is sufficiently obvious.

EXAMPLES FOR PRACTICE.

1. Given $3x=8y-16$, to find the least values of x and y in whole numbers. Ans. $x=8$, $y=5$.

2. Given $14x=5y+7$, to find the least values of x and y in whole numbers. Ans. $x=3$, $y=7$.

3. Given $27x=1600-16y$, to find the least values of x and y in whole numbers. Ans. $x=48$, $y=19$.

4. It is required to divide 100 into two such parts, that one of them may be divisible by 7, and the other by 11. Ans. The only parts are 56 and 44.

5. Given $9x+13y=2000$, to find the greatest value of x and the least value of y in whole numbers.

Ans. $x=215$, $y=5$.

6. Given $11x+5y=254$, to find all the possible values of x and y in whole numbers.

Ans. $x=19, 14, 9, 4$; $y=9, 20, 31, 42$.

7. Given $17x+19y+21z=400$, to find all the answers in whole numbers which the question admits of.

Ans. 10 different answers.

8. Given $5x+7y+11z=224$, to find all the possible values of x y, and z, in whole positive numbers.

Ans. The number of answers is 59.

9. It is required to find in how many different ways it is possible to pay 20*l*. in half-guineas and half-crowns, without using any other sort of coin?

Ans. 7. different ways.

10. I owe my friend a shilling, and have nothing about me but guineas, and he has nothing but louis-d'ors; how must I contrive to acquit myself of the debt, the louis being valued at 17*s*. a piece, and the guineas at 21*s*.?

Ans. I must give him 13 guineas, and he must give me 16 louis.

11. How many gallons of British spirits, at 12*s*., 15*s*., and 18*s*. a gallon, must a rectifier of compounds take to make a mixture of 1000 gallons, that shall be worth 17*s*. a gallon?

Ans. $111\frac{1}{9}$ at 12*s*., $111\frac{1}{9}$ at 15*s*., and $777\frac{7}{9}$ at 18*s*.

PROBLEM II.

To find such a whole number, as, being divided by other given numbers, shall leave given remainders.

RULE.

1. Call the number that is to be determined x, the numbers by which it is to be divided a, b, c, *&c.*, and the given remainders f, g, h, *&c.*

2. Subtract each of the remainders from x, and divide the differences by a, b, c, *&c.*, and there will arise

$$\frac{x-f}{a}, \frac{x-g}{b}, \frac{x-h}{c}, \&c., = \text{whole numbers.}$$

3. Put the first of these fractions $\frac{x-f}{a}=p$, and substitute the value of x, as found in terms of p, from this equation, in the place of x in the second fraction.

4. Find the least value of p in this second fraction, by the last problem, which put $=r$, and substitute the value of x, as found in terms of r, in the place of x in the third fraction.

Find, in like manner, the least value of r, in this third fraction, which put $=s$; and substitute the value of x, as found in terms of s, in the fourth fraction, as before.

Proceed in the same way with the next following fraction, and so on, to the last; when the value of x, thus determined, will give the whole number required.

EXAMPLES.

1. It is required to find the least whole number, which, being divided by 17, shall leave a remainder of 7, and when divided by 26, shall leave a remainder of 13.

Let $x=$ the number required.

Then $\frac{x-7}{17}$ and $\frac{x-13}{26}=$ whole numbers.

And, putting $\frac{x-7}{17}=p$, we shall have $x=17p+7$.

Which value of x, being substituted in the second fraction, gives $\frac{17p+7-13}{26}=\frac{17p-6}{26}=wh.$

But it is obvious that $\frac{26p}{26}$ is also $=wh.$

And consequently $\frac{26p}{26}-\frac{17p-6}{26}=\frac{9p+6}{26}=wh.$

Or $\frac{9p+6}{26}\times 3=\frac{27p+18}{26}=p+\frac{p+18}{26}=wh.$

Where, by rejecting p, there remains $\frac{p+18}{26}=wh.=r.$

Therefore $p=26r-18$;

Whence, if r be taken $=1$, we shall have $p=8$.

And consequently $x=17p+7=17\times 8+7=143$, the number sought.

2. It is required to find the least whole number, which, being divided by 11, 19, and 29, shall leave the remainders 3, 5, and 10, respectively.

Let $x=$ the number required.

Then $\frac{x-3}{11}$, $\frac{x-5}{19}$, and $\frac{x-10}{29}=$ whole numbers.

And, putting $\frac{x-3}{11}=p$, we shall have $x=11p+3$.

Which value of x, being substituted in the second fraction, gives $\frac{11p-2}{19}=wh.$

Or $\frac{11p-2}{19}\times 2=\frac{22p-4}{19}=p+\frac{3p-4}{19}=wh.$

And, by rejecting p, there will remain $\frac{3p-4}{19}=wh.$

Also by mult^n $\frac{3p-4}{19}\times 6=\frac{18p-24}{19}=\frac{18p-5}{19}-1=wh.$

Or, by rejecting the 1, $\frac{18p-5}{19}=wh.$

But $\frac{19p}{19}$ is likewise $=wh.$

Whence $\frac{19p}{19}-\frac{18p-5}{19}=\frac{p+5}{19}=wh$, which put $=r$.

Then we shall have

$p=19r-5$, and $x=11(19r-5)+3=209r-52.$

And if this value be substituted for x in the third fraction, there will arise

$$\frac{209r-62}{29}=7r-2+\frac{6r-4}{29}=wh.$$

Or, by neglecting $7r-2$, we shall have the remaining part of the expression $\frac{6r-4}{29}=wh.$

But, by multiplication,

$$\frac{6r-4}{29}\times 5=\frac{30r-20}{29}=r+\frac{r-20}{29}=wh.$$

Or, by rejecting r, there will remain $\frac{r-20}{29}=wh.$ which put $=s$.

Then $r=29s+20$; where, by taking $s=0$, we shall have $r=20$.

And consequently

$x=209r-52=209\times 20-52=4128$,

which is the number required.

3. To find a number, which, being divided by 6, shall leave the remainder 2, and when divided by 13, shall leave the remainder 3. Ans. 68

4. It is required to find a number, which, being divided by 7, shall leave 5 for a remainder, and if divided by 9, the remainder shall be 2. Ans. 47

5. It is required to find the least whole number, which, being divided by 39, shall leave the remainder 16, and when divided by 56, the remainder shall be 27. Ans. 1147

6. It is required to find the least whole number, which, being divided by 7, 8, and 9, respectively, shall leave the remainders 5, 7, and, 8. Ans. 1727

7. It is required to find the least whole number, which being divided by each of the nine digits, 1, 2, 3, 4, 5, 6, 7, 8, 9, shall leave no remainders. Ans. 2520

8. A person receiving a box of oranges, observed, that, when he told them out by 2, 3, 4, 5, and 6 at a time, he had none remaining; but when he told them out by 7 at a time, there remained 5; how many oranges were there in the box? Ans. 180

OF THE

DIOPHANTINE ANALYSIS.

This branch of Algebra, which is so called from its inventor, Diophantus, a Greek mathematician of Alexandria in Egypt, who flourished in or about the fourth century after Christ, relates chiefly to the finding of square and cube numbers, or to the rendering certain compound expressions free from surds; the method of doing which is by making such substitutions for the unknown quantity, as will reduce the resulting

76 6932
45 4873
31 2059

equation to a simple one, and then finding the value of that quantity in terms of the rest. (*p*)

These questions are so curious and abstruse, that nothing less than the most refined algebra, applied with the greatest skill and judgment can surmount the difficulties which attend them. And in this respect, no one perhaps, has ever excelled DIOPHANTUS, or discovered greater knowledge of the extent and resources of the analytic art.

When we consider his work with attention, we are

(*p*) That DIOPHANTUS was not the inventor of algebra, as has been generally imagined, is obvious; since his method of applying it is such, as could only have been used in an advanced state of the science: besides which, he no where speaks of the fundamental rules and principles, as an inventor certainly would have done, but treats of it as an art already sufficiently known; and seems to intend, not so much to teach it, as to cultivate and improve it, by solving such questions as, before his time, had been thought too difficult to be surmounted.

It is highly probable, therefore, that Algebra was known among the Greeks, long before the time of DIOPHANTUS; but that the works of preceding writers had been destroyed by the ravages of time, or the depredations of war and barbarism.

His *Arithmetical Questions*, out of which a considerable part of these problems are collected, consisted originally of thirteen books: but the first six only are now extant; the best edition of which is that published at Paris, by BACHET, in the year 1670, with Notes by FERMAT: in which work, the subject is so skilfully handled, that the moderns, notwithstanding their other improvements, have been able to do little more than explain and illustrate his method. Those who have succeeded best in this respect, are VIETA, KERSEY, DE BILLY, OZANAM, PRESTET, SAUNDERSON, FERMAT, and EULER; the last of whom in particular has amplified and illustrated the Diophantine Algebra in as clear and satisfactory a manner as the subject seems to admit of.

The reader, who may be desirous of farther information on this interesting subject, will find a methodical abstract of the several methods made use of by these writers, with a variety of examples to illustrate them, in the first and second volumes of my *Treatise on Algebra*, before mentioned.

at a loss which to admire most, his singular sagacity, and the peculiar artifices he employs, in forming such positions as the nature of the problems required, or the more than ordinary subtilty of his reasoning upon them.

Every particular question puts us upon a new way of thinking, and furnishes a fresh vein of analytical treasure, which cannot but prove highly useful to the mind in conducting it through other difficulties of this kind, whenever they may occur, as well as in enabling it to encounter, more readily, those that may arise in subjects of a different nature.

The following method of resolving these questions will be found of considerable service; but no general rule can be given, that will suit all cases; and therefore the solution must often be left to the ingenuity and skill of the learner.

RULE.

1. Put for the root of the square or cube required, one or more letters such that when they are involved, either the given number, or the highest power of the unknown quantity, may vanish from the equation; then if the unknown quantity be only of one dimension, the problem may be solved by reducing the equation.

2. But if the unknown quantity be still a square, or a higher power, some other new letters must be assumed to denote the root; with which proceed as before; and so on, till the unknown quantity is only of one dimension; when, from this, all the rest may be determined.

EXAMPLES.

1. To divide a given square number (100) into two such parts, that each of them may be a square number. (*q*)

(*q*) If $x-10$ had been made the side of the second square, in the following solution of this question, instead of $2x-10$, the

Let x^2 be one of the parts; then $100-x^2$ will be the other part; which is also to be a square number.

Assume the side of this second square $=2x-10$,

Then will $100-x^2=(2x-10)^2=4x^2-40x+100$;

And, consequently, by reduction, $x=8$, and $2x-10=6$,

Therefore 64 and 36 are the parts required.

Or the same may be done generally, thus:

Let a^2 = given square number, x^2 = one of its parts, and a^2-x^2 = the other; which is also to be a square number.

Assume the side of this second square $=rx-a$,
Then will $a^2-x^2=(rx-a)^2=r^2x^2-2arx+a^2$;

And, by reduction, $x=\frac{2ar}{r^2+1}$, and $rx-a=\frac{2ar^2}{r^2+1}-a$

$=\frac{2ar^2}{r^2+1}-\frac{ar^2-a}{r^2+1}=\frac{ar^2-a}{r^2+1}$ = side of the second square.

Therefore $\left(\frac{2ar}{r^2+1}\right)^2$ and $\left(\frac{ar^2-a}{r^2+1}\right)^2$ are the parts required; where a and r may be any whole numbers, taken at pleasure, provided r be greater, than 1. (r)

2. To divide a given number (13) consisting of two known square numbers (9 and 4) into two other square numbers.

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equation would have been $x^2-20x+100=100-x^2$; in which case x, the side of the first square, would have been found $=10$, and $x-10$, or the side of the second square $=0$; for which reason the substitution $x-10$ was avoided; but $3x-10$, $4x-10$, or any other quantity of the same kind, would have succeeded as well as the former, though the results would have been less simple.

(r) To this we may add the following useful property.

If s and r be any two unequal numbers, of which s is the

For the side of the first square sought, put $rx-3$; and for the side of the second, $sx-2$; r being the greater number, and s the less.

Then will $(rx-3)^2+(sx-2)^2=(r^2x^2-6rx+9)+(s^2x^2-4sx+4)=(r^2+s^2)x^2-(6r+4s)x+13=13$, or $(r^2+s^2)x^2=(6r+4s)x$.

From which last equation, we have, $x=\dfrac{6r+4s}{r^2+s^2}$.

Whence $rx-3=\dfrac{6r^2+4rs}{r^2+s^2}-3=\dfrac{3r^2+4rs-3s^2}{r^2+s^2}=$ side of the first square sought.

And, $sx-2=\dfrac{6rs+4s^2}{r^2+s^2}-2=\dfrac{6rs-2r^2+2s^2}{r^2+s^2}=$ side of the second. (s)

So that if r be taken $=2$, and $s=1$, we shall have

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greater, it can then be readily shown, from the nature of the problem, that $2rs$, s^2-r^2 and s^2+r^2 will be the perpendicular, base and hypothenuse of a right-angled triangle.

From which expressions, two square numbers may be found, whose sum or difference shall be square numbers; for $(2rs)^2+(s^2-r^2)^2=(s^2+r^2)^2$, and $(s^2+r^2)^2-(2rs)^2=(s^2-r^2)^2$, or $(s^2+r^2)^2-(s^2-r^2)^2=(2rs)^2$; where s and r may be any numbers whatever.

(s) This question is considered by DIOPHANTUS as a very important one, being made the foundation of many of his other problems; it may be observed, that in the solution of it, given above, the values of r and s may be taken at pleasure, provided the ratio of them be not the same as that of 3 (a) to 2 (b); the reason of which restriction is, that if r and s were so taken, the sides of the squares sought would come out the same as the sides of the known squares which compose the given number, and therefore the operation would be useless.

The excellent old KERSEY, after amplifying and illustrating this problem in a variety of ways, concludes his chapter thus: "For a demonstration of the reverse of this *rare* speculation, see ANDERSONUS, Theorem 2, of VIETA's *mysterious Doctrine of Angular Sections*; and likewise HERIGONIUS, at the latter end of the first *tome* of his *Cursus Mathematicus*.

$\frac{3r^2+4rs-3s^2}{r^2+s^2}=\frac{17}{5}$, and $\frac{6rs-2r^2+2s^2}{r^2+s^2}=\frac{6}{5}$ for the sides of the squares, in numbers, as was required.

And if a^2+b^2 be put equal to the number to be divided, the general solution may be obtained in the same way.

3. To find two square numbers, whose difference shall be equal to any given number.

Let the difference d be resolved into any two unequal factors a and b; a being the greater, and b the less.

Also put x for the side of the less square sought, and $x+b$ for the side of the greater.

Then $(x+b)^2-x^2=x^2+2bx+b^2-x^2=2bx+b^2=d$ $=ab$, by the question.

And if this be divided by b, we shall have $2x+b=a$.

Whence, $x=\frac{a-b}{2}=$the side of the least square sought,

and $x+b=\frac{a-b}{2}+b=\frac{a+b}{2}=$ side of the greater.

So that taking $d=60$, and $a\times b=2\times 30$, we shall have $\frac{30-2}{2}=14$, and $\frac{30+2}{2}=16$.

Whence $(14)^2=196$, and $(16)^2=256$, for the squares in numbers; and so for any difference or factors whatever.

4. To find two numbers such, that if either of them be added to the square of the other, the sum shall be a square number.

Let the numbers sought be x and y.

Then $x^2+y=\square$, and $y^2+x=\square$.

And, if $r-x$ be assumed for the side of the first square x^2+y, we shall have $x^2+y=r^2-2rx+x^2$, or cancelling x^2 on each side of the equation, $y=r^2-2rx$.

Therefore, by reduction, $2rx=r^2-y$, or $x=\frac{r^2-y}{2r}$.

Again, if $y+s$ be assumed for the side of the second square, we shall have $y^2+\frac{r^2-y}{2r}=(y+s)^2=y^2+2sy+s^2$.

Whence also $\frac{r^2-y}{2r}=2sy+s^2$, or $r^2-y=4rsy+2rs^2$.

And consequently, by transposition and division, we shall have $y=\frac{r^2-2rs^2}{4rs+1}$, and $x=\frac{r^2+y}{2r}=\frac{2r^2s+s^2}{4rs+1}$.

So that $\frac{r^2-2rs^2}{4rs+1}$ and $\frac{2r^2s+s^2}{4rs+1}$ are the numbers required; where r and s may be taken at pleasure, provided r be greater than $2s^2$.

5. To find two numbers such, that their sum and difference shall be both square numbers.

Let x and x^2-x be the two numbers sought.

Then, since their sum is evidently a square number, one of the conditions of the question is fulfilled.

There remains, therefore, only their difference x^2-2x to be made a square.

And, if for the side of this square there be put $x-r$, we shall have $x^2-2rx+r^2=x^2-2x$, or $2rx-2x=r^2$.

Whence $x=\frac{r^2}{2r-2}$, and $x^2-x=\left(\frac{r^2}{2r-2}\right)^2-\frac{r^2}{2r-2}$.

So that $\frac{r^2}{2r-2}$ and $\left(\frac{r^2}{2r-2}\right)^2-\frac{r^2}{2r-2}$ are the numbers required, where r may be taken at pleasure, provided it be greater than 1.

6. To find three numbers such, that not only the sum of all three of them, but also the sum of every two, shall be a square number.

Let $4x$, x^2-4x and $2x+1$, be the three numbers sought.

Then $(4x)+(x^2-4x)=x^2$, $(x^2-4x)+(2x+1)=x^2-2x+1$, and $(4x+x^2-4x+2x+1)=x^2+2x+1$, are evidently squares.

And, therefore, three of the conditions mentioned in the questions are fulfilled.

Whence it remains only to make the quantity $(4x)+(2x+1)$, or $6x+1=$to a square.

Let, therefore, $6x+1=a^2$; and we shall have, by transposition and division, $x=\frac{a^2-1}{6}$.

And, consequently, $\frac{4a^2-4}{6}$, $\left(\frac{a^2-1}{6}\right)^2-\frac{4a^2-4}{6}$, and $\frac{2a^2-2}{6}+1$; or $\frac{2a^2-2}{3}$, $\frac{a^4-26a^2+25}{36}$, and $\frac{a^2+2}{3}$ are the numbers required; where a may be any number taken at pleasure, provided it be greater than 5.

7. To find three square numbers, such that the sum of every two of them shall be a square number. (*t*)

Let x^2, y^2, and z^2, be the numbers sought;

Then $x^2+z^2=\square$, y^2+z^2, $=\square$, and $x^2+y^2=\square$.

Or $\frac{x^2}{z^2}+1=\square$, $\frac{y^2}{z^2}+1=\square$, and $\frac{x^2}{z^2}+\frac{y^2}{z^2}=\square$.

And, by putting $\frac{x}{z}=\frac{s^2-1}{2s}$, and $\frac{y}{z}=\frac{r^2-1}{2r}$, we shall

(*t*) This question, like many others here proposed, is capable of a great variety of answers; but the least roots, which have yet been found, in whole numbers, are 44, 117, and 240. These were first given by SAUNDERSON, in Vol. II. of his *Algebra*; and are to be found also in EULER'S *Algebra*, English Translation, Vol. II., which is a work abounding with a great variety of particulars relating to the more abstruse parts of the Diophantine analysis.

have $\frac{x^2}{z^2}+1=\frac{s^4+2s^2+1}{4s^2}$, and $\frac{y^2}{z^2}+1=\frac{r^4+2r^2+1}{4r^2}$, which are both evidently squares; and therefore it only remains to make $\frac{x^2}{z^2}+\frac{y^2}{z^2}=$ square number.

$$\text{But}\ \frac{x^2}{z^2}+\frac{y^2}{z^2}=\left(\frac{s^2-1}{2s}\right)^2+\left(\frac{r^2-1}{2r}\right)^2=\frac{(s^2-1)^2}{4s^2}+\frac{(r^2-1)^2}{4r^2}$$
$$=\frac{r^2\times(s^2-1)^2+s^2\times(r^2-1)^2}{4r^2s^2}=\text{ square number.}$$

Or, by rejecting $4r^2s^2$, $r^2\times(s^2-1)^2+s^2\times(r^2-1)^2=r^2\times(s+1)^2\times(s-1)^2+s^2\times(r+1)^2\times(r-1)^2=$ to a square number.

And, therefore, by making $r-1=s+1$, or $r=s+2$, we shall have $(s+2)^2\times(s+1)^2\times(s-1)^2+s^2\times(s+3)^2\times(s+1)^2=$ to a square number.

Or $(s+2)^2\times(s-1)^2+s^2\times(s+3)^2=2s^4+8s^3+6s^2-4s+4=$ to a square number.

Hence, in order to resolve this expression, let its root be assumed $=\frac{5}{4}s^2-s+2$,

Then, $2s^4+8s^3+6s^2-4s+4=(\frac{5}{4}s^2-s+2)^2=\frac{25}{16}s^4-\frac{5}{2}s^3+5s^2+s^2-4s+4$; or $2s^4+8s^3=\frac{25}{16}s^4-\frac{5}{2}s^3$; or $2s+8=\frac{25}{16}s-\frac{5}{2}$.

From which we have $s=-24$, and $r=-22$.

$$\text{And, } \frac{x}{z}=\frac{s^2-1}{2s}=-\frac{575}{48},\ \text{and } \frac{y}{z}=\frac{r^2-1}{2r}=-\frac{483}{44};$$

$$\text{Or } x=-\frac{575z}{48}\ \text{and } y=-\frac{483z}{44}.$$

Wherefore, to obtain the answer in whole numbers, let z be taken $=528$, and we shall have $x=-6325$, and $y=-5796$.

And, consequently, 528, 5796 and 6325, are the roots of the squares required.

EXAMPLES FOR PRACTICE.

1. It is required to find a number x such, that $x+1$ and $x-1$ shall be both square numbers. (u) Ans. $x=\frac{4}{5}$.

2. It is required to find a number x such, that $x+128$ and $x+192$ shall be both squares. Ans. $x=97$.

3. It is required to find a number such, that x^2+x and x^2-x shall be both squares. Ans. $\frac{25}{24}$.

4. It is required to find two numbers such, that if each of them be added to their product, the sums shall be both squares. Ans. $\frac{2}{3}$ and $\frac{5}{3}$.

5. It is required to find three square numbers in arithmetical progression. Ans. 1, 25, and 49.

6. To find three whole numbers in arithmetical progression, such that the sum of every two of them shall be a square number. Ans. 482, 3362, and 6242.

7. To find three numbers, such that, if to the square of each the sum of the other two be added, the three sums shall be all squares. Ans. 1, $\frac{8}{3}$ and $\frac{16}{3}$.

8. To find two numbers in proportion as 8 is to 15, and such that the sum of their squares shall be a square number. Ans. 576 and 1080.

9. To find two numbers such, that if the square of each be added to their product, the sums shall be both squares. Ans. 9 and 16.

10. To find two whole numbers such, that the sum or difference of their squares, when diminished by unity, shall be a square. Ans. 8 and 9.

11. It is required to resolve 4225, which is the square of 65, into as many other integral squares as the question admits of.

Ans. 16^2+63^2, 56^2+33^2, 60^2+25^2 and 52^2+39^2.

(u) The answers to many of the questions here given, cannot be found in whole numbers.

12. To find three numbers in geometrical proportion, such that each of them, when increased by a given number (19), shall be square numbers.

Ans. 81, $\frac{5}{4}$ and $\frac{25}{1296}$.

13. To find two numbers, such that if their product be added to the sum of their squares, the result shall be a square number. Ans. 5 and 3, 8 and 7, 16 and 5, *&c.*

14. To find three whole numbers, such that if to the square of each the product of the other two be added, the three sums shall be all squares.

Ans. 9, 73 and 328.

15. It is required to find three square numbers, such that their sum when added to each of their three roots, shall be all square numbers.

Ans. $\frac{4418}{62920}$, $\frac{13254}{62920}$, and $\frac{19881}{62920}$=roots required.

16. To find three numbers in geometrical progression, such that if the mean be added to each of the extremes, the sums, in both cases, shall be squares.

Ans. 5, 20 and 80.

17. To find two numbers, such that not only each of them, but also their sum and their difference, when increased by unity, shall be all square numbers.

Ans. 1368 and 840.

18. To find three numbers, such that whether their sum be added to, or subtracted from, the square of each of them, the numbers thence arising shall be all squares. Ans. $\frac{406}{96}$, $\frac{518}{96}$ and $\frac{701}{96}$.

19. It is required to find three square numbers, such that the sum of their squares shall also be a square number. Ans. 9, 16 and $\frac{144}{25}$.

20. It is required to find three square numbers, such that the difference of every two of them shall be a square number. Ans. 485809, 34225 and 23409.

21. It is required to divide any given cube number (8) into three other cube numbers. Ans. 1, $\frac{64}{27}$ and $\frac{125}{27}$.

22. To find three square numbers, such that the difference between every two of them, and the third shall be a square number. Ans. 149^2, 241^2, and 269^2.

23. To find three cube numbers, such that if from each of them a given number (1) be subtracted, the sum of the remainders shall be a square number.

Ans. $\frac{4913}{3375}$, $\frac{21952}{3375}$ and 8.

OF THE

SUMMATION AND INTERPOLATION OF INFINITE SERIES.

THE doctrine of Infinite Series is a subject which has engaged the attention of the greatest mathematicians, both of ancient and modern times; and, when taken in its whole extent, is, perhaps, one of the most abstruse and difficult branches of abstract mathematics.

To find the sum of a series, the number of the terms of which is inexhaustible or infinite, has been regarded, by some, as a paradox, or a thing impossible to be done; but this difficulty will be easily removed, by considering that every finite magnitude whatever is divisible *ad infinitum*, or consists of an indefinite number of parts, the aggregate, or sum, of which, is equal to the quantity first proposed.

A number actually infinite is, indeed, a plain contradiction to all our ideas; for any number that we can possibly conceive, or of which we have any notion, must always be determinate and finite; being such that a greater may still be assigned, and a greater after this; and so on, without a possibility of ever coming to an end of the increase or addition.

This inexhaustibility, therefore, in the nature of numbers, is all that we can distinctly comprehend by their infinity; for though we can easily conceive that a finite quantity may become greater and greater without end, yet we are not, by that means, enabled to form any notion of the *ultimatum*, or last magnitude, which is incapable of farther augmentation.

Hence, we cannot apply to an infinite series the common notion of a sum, or of a collection of several particular numbers, which are joined aud added together, one after another; as this supposes that each of the numbers, composing that sum, is known and determined. But as every series generally observes some regular law, and continually approaches towards a term, or limit, we can easily conceive it to be a whole of its own kind, and that it must have a certain real value, whether that value be determinable or not.

Thus, in many series, a number is assignable, beyond which no number of its terms can ever reach, or, indeed, be ever perfectly equal to it; but yet may approach towards it, in such a manner, as to differ from it by less than any quantity that can be named. So that we may justly call this the value or sum of the series; not as being a number found by the common method of addition, but such a limitation of the value of the series, taken in all its infinite capacity, that, if it were possible to add all the terms together, one after another, the sum would be equal to that number.

In other series, on the contrary, the aggregate, or value of the several terms, taken collectively, has no limitation; which state of it may be expressed, by saying, that the sum of the series is infinitely great; or, that it has no determinate or assignable value, but may be carried on to such a length, that its sum shall exceed any given number whatever.

Thus, as an illustration of the first of these cases, it

may be observed, that, if r be the ratio, g the greatest term, and l the least of any decreasing geometric series, the sum, according to the common rule, will be $(rg-l) \div (r-1)$: and if we suppose the less extreme, l, to be diminished till it becomes $=0$, the sum of the whole series will be $rg \div (r-1)$: for it is demonstrable, that the sum of no assignable number of terms of the series can ever be equal to that quotient; and yet no number less than it will ever be equal to the value of the series.

Whatever consequences, therefore, follow from the supposition of $rg \div (r-1)$ being the true and adequate value of the series, taken in all its infinite capacity, as if all the parts were actually determined, and added together, no assignable error can possibly arise from them, in any operation, or demonstration, where the sum is used in that sense; because, if it should be said that the series exceeds that value, it can be proved, that this excess must be less than any assignable difference; which is in effect no difference at all; whence the supposed error cannot exist; and consequently $rg \div (r-1)$ may be looked upon as expressing the true value of the series, continued to infinity.

We are also, farther satisfied of the reasonableness of this doctrine, by finding, in fact, that a finite quantity is frequently convertible into an infinite series, as appears in the case of circulating decimals. Thus two-thirds expressed decimally, is $\frac{2}{3} = .66666 \ldots = \frac{6}{10} + \frac{6}{100} + \frac{6}{1000} + \frac{6}{10000} +$ &c., continued *ad infinitum*. But this is a geometric series, the first term of which is $\frac{6}{10}$, and the ratio $\frac{1}{10}$; and therefore the sum of all its terms, continued to infinity, will evidently be equal to $\frac{2}{3}$, or the number from which it was originally derived. And the same may be shown of many other series, and of all circulating decimals in general.

With respect to the processes by which the sum-

mation of various kinds of infinite series are usually obtained, one of the principal is by the method of differences, pointed out and illustrated in Prob. IV. next following.

Another method is that first employed by JAMES and JOHN BERNOULLI, which consists in resolving the given series into others of a different kind, of which the summation can be determined; or by subtracting from an assumed series, the value of which is denoted by the same series, deprived of some of its first terms; in which case a new series will arise, whose sum will be known.

A third method, which is that of DEMOIVRE, consists in putting the sum of the series $=s$, and multiplying each side of the equation by some binomial or trinomial expression, which involves the powers of the unknown quantity x, and certain known co-efficients; then, taking x, after the process is performed, of such a value that the assumed binomial, *&c.*, shall become $=0$, and transposing some of the first terms; when a series will arise, the sum of which will be known, as before.

Each of which methods, modified so as to render it more commodious in practice, together with several other artifices for the same purpose, will be found sufficiently elucidated in the miscellaneous questions succeeding the following problems.

PROBLEM I.

Any series being given to find its several orders of differences.

RULE.

1. Take the first term from the second, the second from the third, the third from the fourth, *&c.*, and the remainders will form a new series, called the *first order of differences.*

2. Take the first term of this last series from the

second, the second from the third, the third from the fourth, *&c.*, and the remainders will form another new series, called the *second order of differences.*

3. Proceed, in the same manner, for the third, fourth, fifth, *&c.*, orders of differences; and so on till they terminate, or are carried as far as may be thought necessary. (x)

EXAMPLES.

1. Required the several orders of differences of the series, $1, 2^2, 3^2, 4^2, 5^2, 6^2$, *&c.*

Here 1,	4,	9,	16,	25,	36,	*&c.*	given series.
	3,	5,	7,	9,	11,	*&c.*	1st diff.
		2,	2,	2,	2,	*&c.*	2d diff.
			0,	0,	0,	*&c.*	3d diff.

2. Required the several orders of differences of the series, $1, 2^3, 3^3, 4^3, 5^3, 6^3$, *&c.*

Here 1,	8,	27,	64,	125,	216,	*&c.*	given series.
	7,	19,	37,	61,	91,	*&c.*	1st diff.
		12,	18,	24,	30,	*&c.*	2d diff.
			6,	6,	6,	*&c.*	3d diff.
				0,	0,	*&c.*	4th diff.

3. Required the several orders of differences of the series, 1, 3, 6, 10, 15, 21, *&c.*

Ans. 1st, 2, 3, 4, 5, *&c.*; 2d, 1, 1, 1, *&c.*

4. Required the several orders of differences of the series 1, 6, 20, 50, 105, 196, *&c.*

Ans. 1st, 5, 12, 30, 45, 91, *&c.*; 2d, 9, 16, 25, 36, *&c.*; 3d, 7, 9, 11, *&c.*; 4th, 2, 2, *&c.*

(x) When the several terms of the series continually increase, the differences will be all positive; but when they decrease, the differences will be negative and positive alternately.

5. Required the several orders of differences of the series $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}$, &c.

PROBLEM II.

Any series, a, b, c, d, e, &c., being given, to find the first term of the nth order of differences.

Let δ stand for the first term of the nth differences.

Then will $a - nb + \frac{n(n-1)}{2}c - \frac{n(n-1)}{2}\frac{(n-2)}{3}d + \frac{n(n-1)(n-2)(n-3)}{2.\ 3.\ 4}e$, &c., to $n+1$ terms $=\delta$, when n is an even number.

And $-a + nb - \frac{n(n-1)}{2}c + \frac{n(n-1)(n-2)}{2\ .\ 3}d - \frac{n(n-1)}{2}\frac{(n-2)(n-3)}{3\ .\ 4}e$, &c., to $n+1$ terms $=\delta$, when n is an odd number. (y)

EXAMPLES.

1. Required the first term of the third order of differences of the series 1, 5, 15, 35, 70, &c.

Here a, b, c, d, e, &c., $=1, 5, 15, 35, 70$, &c., respectively, and $n=3$.

Whence $-a + nb - \frac{n(n-1)}{2}c + \frac{n(n-1)}{2}\frac{(n-2)}{3}d = -a$

(y) When the terms of the several orders of differences happen to be very great, it will be more convenient to take the logarithms of the quantities concerned, whose differences will be smaller; and, when the operation is finished, the quantity answering to the last logarithm may be easily found.

$+3b-3c+d=-1+15-45+35=4=$ the first term required.

2. Required the first term of the fourth order of differences of the series 1, 8, 27, 64, 125, &c.

Here a, b, c, d, e, &c., $=1, 8, 27, 64, 125$, &c., respectively, and $n=4$.

Whence $a-nb+\frac{n(n-1)}{2}c-\frac{n(n-1)(n-2)}{2\ \ 3}d+$
$\frac{n(n-1)(n-2)(n-3)}{2\ \ 3\ \ 4}e=a-4b+6c-4d+e=1-32+$
$162-256+125=0$; so that the first term of the fourth order is 0.

3. Required the first term of the eighth order of differences of the series 1, 3, 9, 27, 81, &c.

Ans. 256

4. Required the first term of the fifth order of differences of the series, $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}$, &c. (z)

Ans. $-\frac{1}{32}$

PROBLEM III.

To find the nth term of the series, a, b, c, d, e, &c., when the differences of any order become at last equal to each other.

RULE.

Let $d', d'', d''', d^{\text{iv}}$, &c., be the first terms of the several orders of differences, found as in the last problem.

(z) The labour, in questions of this kind, may be often abridged, by putting ciphers for some of the terms at the beginning of the series; by which means several of the differences will be equal to 0, and the answer, on that account, obtained in fewer terms.

Then will $a+\frac{n-1}{1}d'+\frac{(n-1)\ (n-2)}{1\ .\ 2}d''+\frac{(n-1)}{1}$ $\frac{(n-2)(n-3)}{2\quad 3}d'''+\frac{(n-1)(n-2)(n-3)(n-4)}{1\ .\ 2\ .\ 3\ .\ 4}d^{\text{iv}}, +\&c.$ $=n$th term required.

EXAMPLES.

1. It is required to find the twelfth term of the series 2, 6, 12, 20, 30, &c.

Here 2,	6,	12,	20,	30,	&c.	given series.
	4,	6,	8,	10,	&c.	1st diff.
		2,	2,	2,	&c.	2d diff.
			0,	0,	&c.	3d diff.

Whence 4 and 2 are the first terms of the differences.

Let, therefore, $4=d'$, $2=d''$, and $n=12$.

Then $a+\frac{n-1}{1}d'+\frac{(n-1)(n-2)}{1\ .\ 2}d''=2+11d'+55d''$ $=2+44+110=156=12$th term, or the answer required.

2. Required the twentieth term of the series, 1, 3, 6, 10, 15, 21, &c.

Here 1,	3,	6,	10,	15,	21,	&c.	given series.
	2,	3,	4,	5,	6,	&c.	1st diff.
		1,	1,	1,	1,	&c.	2d diff.
			0,	0,	0,	&c.	3d diff.

Where 2 and 1 are the first terms of the differences.

Let, therefore, $2=d'$, $1=d''$, and $n=20$.

Then $a+\frac{n-1}{1}d'+\frac{(n-1)(n-2)}{1\ .\ 2}d''=1+19d'+171d''$ $=1+38+171=210=20$th term required.

3. Required the fifteenth term of the series, 1, 4, 9, 16, 25, 36, &c. Ans. 225

4. Required the twentieth term of the series, 1, 8, 27, 64, 125, &c.

5. It is required to find the thirtieth term of the series, $1, \frac{1}{3}, \frac{1}{6}, \frac{1}{10}, \frac{1}{15}, \frac{1}{21}, \frac{1}{28}$, *&c.*

PROBLEM IV. (*a*)

To find the sum of n terms of the series, a, b, c, d, e, *&c.*, when the differences of any order become at last equal to each other.

RULE.

Let d', d'', d''', d^{iv}, *&c.* be the first terms of the several orders of differences.

Then will $na + \frac{n(n-1)}{2}d + \frac{n(n-1)(n-2)}{2\,.\,3}d'' +, \frac{n(n-1)(n-2)(n-3)}{2\,.\,3\,.\,4}d''' + \frac{n(n-1)(n-2)(n-3)(n-4)}{2\,.\,3\,.\,4\,.\,5}d^{iv}$. *&c.*, $=$ to the sum of n terms of the series.

EXAMPLES.

1. Required the sum of n terms of the series, 1, 2, 3, 4, 5, 6, *&c.*

Here 1,	2,	3,	4,	5,	6,	*&c.* given series.
	1,	1,	1,	1,	1,	*&c.* 1st diff.
		0,	0,	0,	0,	*&c.* 2d diff.

where 1 and 0 are the first terms of the differences.

Let, therefore, $a = 1$, $d' = 1$, and $d'' = 0$.

Then will $na + n.\frac{n-1}{2}d' = n + \frac{n^2-n}{2} = \frac{n^2+n}{2} = \frac{n(n+1)}{2}$ the sum of n terms, as required.

(*a*) When the differences in this or the former rule are finally $=0$, any term, or the sum of any number of the terms may be accurately determined; but if the differences do not vanish, the result is only an approximation; which, however, may be often very usefully applied in resolving various questions that may occur in this branch of the subject; and which will become continually nearer the truth as the differences diminish.

2. Required the sum of n terms of the series, 1^2, 2^2, 3^2, 4^2, 5^2, &c., or 1, 4, 9, 16, 25, &c.

Here 1,	4,	9,	16,	25;	&c.	given series.
	3,	5,	7,	9,	&c.	1st diff.
		2,	2,	2,	&c.	2d diff.
			0,	0,	&c.	3d diff.

where 3 and 2 are the first terms of the differences.

Let, therefore, $a=1$, $d'=3$, and $d''=2$.

Then will $na+\frac{n(n-1)}{2}d'+\frac{n(n-1)(n-2)}{2\,.\,3}d=n+$
$\frac{3n(n-1)}{2}+\frac{2n(n-1)(n-2)}{2\,.\,3}=n+\frac{3n^2-3n}{2}+$
$\frac{n^3-3n^2+2n}{3}=\frac{2n^3+3n^2+n}{6}=\frac{n\times(n+1)\times(2n+1)}{6}=$ Ans.

3. Required the sum of n terms of the series, 1^3, 2^3, 3^3, 4^3, 5^3, &c., or 1, 8, 27, 64, 125, &c.

Here 1,	8,	27,	64,	125,	&c.	given series.
	7,	19,	37,	61,	&c.	1st diff.
		12,	18,	24,	&c.	2d diff.
			6,	6,	&c.	3d. diff.
				0,	&c.	4th diff.

where the first terms of the differences are 7, 12, and 6.

Let, therefore, $a=1$, $d'=7$, $d''=12$, and $d'''=6$.

Then will $na+\frac{n(n-1)}{2}d'+\frac{n(n-1)(n-2)}{2\,.\,3}d''+\frac{n(n-1)}{2}$
$\frac{(n-2)(n-3)}{3\,.\,4}d'''=n+\frac{7n(n-1)}{2}+\frac{12n(n-1)(n-2)}{2\,.\,3}+$
$\frac{6n(n-1)(n-2)(n-3)}{2\,.\,3\,.\,4}=n+\frac{7n^2-7n}{2}+2n^3-6n^2+4n+$
$\frac{n^4-6n^3+11n^2-6n}{4}=\frac{4n}{4}+\frac{14n^2-14n}{4}+\frac{8n^3-24n^2+16n}{4}$

$$+\frac{n^4-6n^3+11n^2-6n}{4}=\frac{n^4+2n^3+n^2}{4}=\frac{n^2(n+1)^2}{4}=\text{sum}$$

of n terms, as required.

4. Required the sum of n terms of the series, 2, 6, 12, 20, 30, *&c.* Ans. $\frac{n\times(n+1)\times(n+2)}{3}$

5. Required the sum of n terms of the series, 1, 3, 6, 10, 15, *&c.* Ans. $\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}$

6. Required the sum of n terms of the series, 1, 4, 10, 20, 35, *&c.* Ans. $\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}\times\frac{n+3}{4}$

7. Required the sum of n terms of the series, 1^4, 2^4, 3^4, 4^4, *&c.*, or 1, 16, 81, 256, *&c.*

Ans. $\frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}$

8. Required the sum of n terms of the series, 1^5, 2^5, 3^5, 4^5, 5^5, *&c.*

Ans. $\frac{n^6}{6}+\frac{n^5}{2}+\frac{5n^4}{12}-\frac{n^2}{12}$

PROBLEM V.

Any series $a, b, c, d, e,$ *&c.* of equidistant terms, being given, to find any intermediate term by interpolation.

RULE.

Let x be the place in the series, of any term z, that is to be interpolated; and d', d'', d''', *&c.* the first terms of the several orders of differences.

Then will $z=a+(x-1)d'+\frac{(x-1)(x-2)}{2}d''+$

$$\frac{(x-1)(x-2)(x-3)}{2\,.\,3}d'''+\frac{(x-1)(x-2)(x-3)(x-4)}{2\,.\,3\,.\,4}d^{iv}+\&c$$

EXAMPLES.

1. Given the logarithmic sines of 1° 0′, 1° 1′, 1° 2′, and 1° 3′, to find the log. sine of 1° 1′ 40″.

Here	1° 0′	1° 1′	1° 2′	1° 3′
Sines	8.2418553	8.2490332	8.2560943	8.2630424
	1st diff.	71779	70611	69481
		2d diff.	−1168	−1130
			3d diff.	38

Therefore 71779, −1168, and 38, are the 1st terms of the differences.

And since 1° 1′ 40″, falls between the second and third terms, and 1′ 40″, $=1\frac{2}{3}$, x will be $=1+1\frac{2}{3}=2\frac{2}{3}=\frac{8}{3}$, $d'=71779$, $d''=-1168$ and $d'''=38$.

Hence $z=a+(x-1)d'+\frac{(x-1)(x-2)}{2}d''+\frac{(x-1)}{1}\frac{(x-2)(x-3)}{2\ .\ 3}=a+\frac{5}{3}d'+\frac{5}{9}d''-\frac{5}{81}d'''=8.2418553+.0119631+.0000694-.0000002=8.2538232=$ sine of 1° 1′ 40″, as was required.

2. Given the series $\frac{1}{50}, \frac{1}{51}, \frac{1}{52}, \frac{1}{53}, \frac{1}{54}$, &c. to find the term which stands in the middle between the two terms $\frac{1}{52}$ and $\frac{1}{53}$. Ans. $\frac{1}{52.46}$

3. Given the natural tangents of 88° 54′, 88° 55′, 88° 56′, 88° 57′, 88° 58′, 88° 59′ to find the tangent of 88° 58′ 11″. Ans. 55.711144

PROBLEM VI.

When the differences of any order of the series, a, b, c, d, e, &c., are very small, or become equal to 0, any intermediate term may be interpolated as follows,

RULE.

Find the value of the unknown quantity in the equation which stands against the given number of terms, in the following table, and it will give the term required. (b)

1. $a-b=0$
2. $a-2b+c=0$
3. $a-3b+3c-d=0$
4. $a-4b+6c-4d+e=0$
5. $a-5b+10c-10d+5e-f=0$
6. $a-6b+15c-20d+15e-6f+g=0$

&c. &c.

Or universally for n terms.

$$a-nb+\frac{n(n-1)}{2}c-\frac{n(n-1)(n-2)}{2\,.\,3}d+\frac{n(n-1)(n-2)(n-3)}{2\,.\,3\,.\,4}e-\&c.=0.$$

EXAMPLES.

1. Given the logarithms of 101, 102, 104, and 105, to find the logarithm of 103.

Here the number of terms are 4.

And against 4, in the table, we have $a-4b+6c-4d+e=0$; or $c=\frac{4\times(b+d)-(a+e)}{6}=$ value of the unknown quantity, or term to be found.

Where taking the logs of 101, 102, 104, and 105, we have

$a=2.0043214$
$b=2.0086002$
$d=2.0170333$
$e=2.0211893$

(b) The more terms are given, in any series of this kind, the more accurately will the equation that is to be used approximate towards the true result, or answer required.

And consequently

$$\begin{array}{r} 4\times(b+d)=16.1025340 \\ a+e=\ \ 4.0255107 \\ \hline 6)12.0770233 \\ \hline 2.0128372 \end{array}=\text{log. of } 103,$$

as required.

2. Given the cube roots of 45, 46, 47, 48, and 49, to find the cube root of 50. Ans. 3.684031.

3. Given the logarithms of 50, 51, 52, 54, 55, and 56, to find the logarithm of 53. Ans. 1.7242758695.

PROMISCUOUS EXAMPLES RELATING TO SERIES.

1. To find the sum (s) of n terms of the series, 1, 2, 3, 4, 5, 6, &c.

Let $1+2+3+4+5+$ &c. . . . . $+n=s$.

Then $n+(n-1)+(n-2)+(n-3)+(n-4)$
$+(n-5)+$ &c. . . . $+1=s$.

Whence, by addition,

$(n+1)+(n+1)+(n+1)+(n+1)+(n+1)+$ &c. . . . to n terms $=2s$.

And consequently $n(n+1)=2s$; or $s=\frac{n(n+1)}{2}=$ sum required.

2. To find the sum (s) of n terms of the series, 1, 3, 5, 7, 9, 11, &c.

Let $1+3+5+7+9$ &c. . . . . $(2n-1)=s$.

Then $(2n-1)+(2n-3)+(2n-5)+$ . . . $+1=S$,

Whence, by addition,

$2n+2n+2n+2n+2n+$ &c. to n terms $=2s$.

And consequently $2n\times n=2s$;

$$\text{Or } s = \frac{2n^2}{2} = n^2 = \text{ sum required.}$$

3. Required the sum (s) of n terms of the series, $a+(a+d)+(a+2d)+(a+3d)+(a+4d)+$ &c.

Let $S=a+(a+d)+(a+2d)+(a+3d)+\ldots+\{a+(n-1)d\}$

Then, by reversing the series,

$S=\{a+(n-1)d\}+\{(a+(n-2)d\}+\{a+(n-3)d\}+\ldots+a.$

Whence, by addition, $\{2a+(nd-d)\}+\{2a+(nd-d)\}+\{2a+(nd-d)\}+$ &c., to n terms.

And consequently $2S=(2a+nd-d)\times n$;

$$\text{Or } s=\{2a+(n-1)d\}\times\frac{n}{2}= \text{ sum required.}$$

Or the same may be determined in a different manner, as follows:

$$a+(a+d)+(a+2d)+(a+3d)+(a+4d) \;\&c.$$

$$= \left|\begin{matrix}(+1+1+1+1+1+\;\&c.)\times a\\(+0+1+2+3+4+\;\&c.)\times d\end{matrix}\right| = s.$$

But n terms of $1+1+1+1+1+$ &c.$=n$.

And n terms of $0+1+2+3+4$ &c.$=\dfrac{n(n-1)}{2}$

$$\text{Whence } s=na+\frac{n(n-1)d}{2}=\{2a+(n-1d)\}\times\frac{n}{2},$$

which is the same answer as before.

4. To find the sum (s) of n terms of the series $1, x, x^2, x^3, x^4, x^5, x^6,$ &c.

Let $1+x+x^2+x^3+x^4,$ &c. . . . . . $x^{n-1}=s$.

Then $x+x^2+x^3+x^4+x^5,$ &c. . . . . $x^n=sx$.

Whence, by subtraction, $x^n-1=sx-s$.

$$\text{Or } s=\frac{x^n-1}{x-1}= \text{ sum required.}$$

And when x is a proper fraction, the sum of the series, continued ad infinitum, may be found in the same manner.

Thus, putting $1+x+x^2+x^3+x^4+x^5$, &c.$=$s.

We shall have $x+x^2+x^3+x^4+x^5$, &c.$=$sx.

And consequently $-1=$s$x-$s ; or s$-$s$x=1$,

Whence s$=\frac{1}{1-x}=$ sum of an infinite number of terms of the series, as was to be found.

5. Required the sum (s) of the circulating decimal .999999 . . . continued ad infinitum.

$$\text{Here } .999999=\frac{9}{10}+\frac{9}{100}+\frac{9}{1000}+\frac{9}{10000}+\&c.$$

$$=9(\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\frac{1}{10000}+\&c.)=\text{s}.$$

$$\text{Or, } \frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\frac{1}{10000}+\&c.=\frac{\text{s}}{9}.$$

$$\text{Whence } 1+\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\&c.=\frac{10\text{s}}{9}.$$

And consequently, by subtraction, $1=\frac{10\text{s}}{9}-\frac{\text{s}}{9}=\frac{9\text{s}}{9}=$s ;

Or s$=1=$ sum of the series,

6. Required the sum (s) of the series $a^2+(a+d)^2+(a+2d)^2+(a+3d)^2+(a+4d)^2+$&c., continued to n terms.

Here

$$a^2=a^2$$
$$(a+d)^2=a^2+2\times1ad+1d^2$$
$$(a+2d)^2=a^2+2\times2ad+4d^2$$

$(a+3d)^2=a^2+2\times 3ad+9d^2$
$(a+4d)^2=a^2+2\times 4ad+16d^2$
&c. &c.

Whence

$$s=\left|\begin{array}{l}\text{Sum of } n \text{ terms of } (1+1+1+1+1+\ \&c.)a^2\\ +\ \ldots\ \text{ditto of } (0+1+2+3+4+\ \&c.)2ad\\ +\ \ldots\ \text{ditto of } (0+1+4+9+16+\ \&c.)d^2\end{array}\right.$$

But n terms of $1+1+1+1+1+$ &c. $=n$.

And of $0+1+2+3+4$ &c. $=\dfrac{n(n-1)}{1\ .\ 2}$

Also of $0+1+4+9+$ &c. $=\dfrac{n(n-1)(2n-1)}{1\ .\ 2\ .\ 3}$

Therefore $s=na^2+n(n-1)ad+\dfrac{n(n-1)(2n-1)}{1\ .\ 2\ .\ 3}d^2=$ the whole sum of the series to n terms.

7. Required the sum (s) of the series $a^3+(a+d)^3+(a+2d)^3+(a+3d)^3+(a+4d)^3+$ &c., continued to n terms.

Here, $a^3=a^3$
$(a+d)^3=a^3+3\times 1a^2d+3\times 1ad^2+1d^3$
$(a+2d)^3=a^3+3\times 2a^2d+3\times 4ad^2+8d^3$
$(a+3d)^3=a^3+3\times 3a^2d+3\times 9ad^2+27d^3$
$(a+4d)^3=a^3+3\times 4a^2d+3\times 16ad^2+64d^3$
$(a+5d)^3=a^3+3\times 5a^2d+3\times 25ad^2+125d^3$
&c. &c.

Whence

$$s=\left|\begin{array}{l}\text{Sum of } n \text{ terms of } (1+1+1+1+1\ \&c.)a^3\\ +\ \ldots\ \text{ditto of } (0+1+2+3+4+\ \&c.)3a^2d\\ +\ \ldots\ \text{ditto of } (0+1+4+9+16+\ \&c.)3ad^2\\ +\ \ldots\ \text{ditto of } (0+1+8+27+64+\ \&c.)d^3\end{array}\right.$$

But n terms of $1+1+1+1+1+1$ &c. $=n$.

Ditto . . . of $0+1+2+3+4$ &c. $=\dfrac{n(n-1)}{1\ .\ 2}$

Ditto . . . of $0+1+4+9+16$ &c. $=\dfrac{n(n-1)(2n-1)}{1\ .\ 2\ .\ 3}$

Ditto . . of $0+1+8+27+64+$ &c. $=\frac{n^4-2n^3+n^2}{4}$

Therefore $s=na^3+\frac{n(n-1)3a^2d}{2}+\frac{n(n-1)(2n-1)3ad^2}{6}+\frac{(n^4-2n^3+n^2)d^3}{4}=$ sum of n terms, as was to be found.

8. Required the sum (s) of n terms of the series $1+3+7+15+31+$ &c.

Here the terms of this series are evidently equal to 1, $(1+2)$, $(1+2+4)$, $(1+2+4+8)$, &c., or to the successive sums of the geometrical series 1, 2, 4, 8, 16, &c.

Let, therefore, $a=1$ and $r=2$, and we shall have $a+ar+ar^2+ar^3+ar^4$ &c. $=1+2+4+8+16$, &c.

But the successive sums of 1, 2, 3, 4, &c. terms of this series are,

$$1.\ \frac{ar-a}{r-1}=(r-1)\times\frac{a}{r-1}$$

$$2.\ \frac{ar^2-a}{r-1}=(r^2-1)\times\frac{a}{r-1}$$

$$3.\ \frac{ar^3-a}{r-1}=(r^3-1)\times\frac{a}{r-1}$$

$$4.\ \frac{ar^4-a}{r-1}=(r^4-1)\times\frac{a}{r-1}$$

&c. &c.

Whence $s=\frac{a}{r-1}\times \Big|$ n terms of $r+r^2+r^3+r^4+$ &c. $-n$ terms of $1+1+1+1+$ &c.

But $1+1+1+1+1+1+1$ &c. $=n$.

And $r+r^2+r^3+r^4+$ &c. $=(r^n-1)\times\frac{r}{r-1}$

Therefore $s=\frac{r(r^n-1)}{r-1}\times\frac{a}{r-1}-n\times\frac{a}{r-1}=2(2^n-1-n)$ $=$ sum required.

9. It is required to find the sum of n terms of the series $\frac{1}{1}+\frac{3}{2}+\frac{7}{4}+\frac{15}{8}+\frac{31}{16}+\frac{63}{32}+$ &c.

Here the terms of this series are the successive sums of the geometrical series $\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+$ &c.

Let, therefore, $a=1$ and $r=2$, then will

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\text{&c.}=a+\frac{a}{r}+\frac{a}{r^2}+\frac{a}{r^3}+\frac{a}{r^4}+\frac{a}{r^5}+\frac{a}{r^6}\text{&c.}$$

But the successive sums of 1, 2, 3, 4, &c., terms of this series are

$$1.\ \frac{(r-1)\times a}{(r-1)\times 1}=(r-1)\times\frac{a}{r-1}$$

$$2.\ \frac{(r^2-1)\times a}{(r-1)\times r}=(r-\frac{1}{r})\times\frac{a}{r-1}$$

$$3.\ \frac{(r^3-1)\times a}{(r-1)\times r^2}=(r-\frac{1}{r^2})\times\frac{a}{r-1}$$

$$4.\ \frac{(r^4-1)\times a}{(r-1)\times r^3}=(r-\frac{1}{r^3})\times\frac{a}{r-1}$$

&c. &c.

Therefore

$$s=\frac{a}{r-1}\times\left|\begin{array}{l} n \text{ terms of } r+r+r+r+r+\text{ &c.} \\ -n \text{ terms of } \frac{1}{1}+\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}+\text{ &c.}\end{array}\right.$$

These being the two series derived from the above expressions,

But $r+r+r+r+r+r+r+$ &c. $=nr$.

And $\frac{1}{1}+\frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}+$ &c. $=\frac{r^n-1}{(r-1)r^{n-1}}$

Whence

$s = \frac{a}{r-1} \times (nr - \frac{r^n - 1}{(r-1)r^{n-1}}) = \frac{(n-1)2^n + 1}{2^{n-1}} =$ sum required.

10. Required the sum (s) of the infinite series of the reciprocals of the triangular numbers $\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} +$ &c., continued to infinity.

Let $\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} +$ &c. *ad infinitum* $= s$.

Or $\frac{1}{1.1} + \frac{1}{1.3} + \frac{1}{2.3} + \frac{1}{2.5} +$ &c. . . . . . $= s$.

Then $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} +$ &c. . . . . $= \frac{s}{2}$.

That is, $(\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{5})$ &c. $= \frac{s}{2}$.

Or, $\left| \begin{array}{l} \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \text{\&c.} \\ - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \text{\&c.} \end{array} \right| = \frac{s}{2}$

Whence $\frac{s}{2} = \frac{1}{1}$; or $s = 2 =$ sum required.

11. It is required to find the sum of n terms of the same series, $\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} +$ &c.

Let $z = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} +$ &c. to $\frac{1}{n}$.

Then $z - \frac{1}{1} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} +$ &c. to $\frac{1}{n}$.

And $z-\frac{1}{1}+\frac{1}{n+1}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+$ &c. to $\frac{1}{n+1}$.

Therefore, by subtracting this from the first, we have

$\frac{1}{1}-\frac{1}{n+1}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}$ &c. to $\frac{1}{n}-\frac{1}{n+1}$.

Or $\frac{n}{n+1}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}$ &c. to $\frac{1}{n(n+1)}$

Whence $\frac{2n}{n+1}=\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+$ &c. to $\frac{2}{n(n+1)}$.

Or $\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+$ &c. to $\frac{2}{n(n+1)}=\frac{2n}{n+1}=$ sum of n terms of the series, as was required.

12. It is required to find the sum of the series, $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\frac{1}{4.5.6}+$ &c. continued to infinity.

Let $z=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+$ &c. ad infinitum.

Then $z-\frac{1}{1}=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+$ &c. by transposition.

And $1=\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+$ &c. by subtraction.

Or $1-\frac{1}{2}=\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\frac{1}{5.6}+$ &c. by transposition.

Whence $\frac{1}{2}=\frac{4}{1.4.3}+\frac{6}{2.9.4}+\frac{8}{3.16.5}+$ &c. by subtraction.

Or $\frac{1}{2}=\frac{2}{1.2.3}+\frac{2}{2.3.4}+\frac{2}{3.4.5}+\frac{2}{4.5.6}+$ &c.

And $\frac{1}{2}\div 2=\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+$ &c.

But $\frac{1}{2}\div 2=\frac{1}{4}$; therefore $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\frac{1}{4.5.}$ $+$ ad infinitum $=\frac{1}{4}$, which is the sum required.

13. It is required to find the sum of n terms of th same series $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\frac{1}{4.5.6}+$ &c.

$$\text{Let } z=\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\text{ \&c. to }\frac{1}{n(n+1)}$$

$$\text{Then } z-\frac{1}{2}=\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\text{ \&c. to }\frac{1}{n(n+1)}.$$

$$\text{And } z-\frac{1}{2}+\frac{1}{(n+1)(n+2}=\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\frac{1}{5.6}+$$ $\frac{1}{6.7}+\frac{1}{7.8}+$&c., continued to $\frac{1}{(n+1)(n+2)}$ terms.

$$\text{Therefore } \frac{1}{2}-\frac{1}{(n+1)(n+2)}=\frac{2}{1.2.3}+\frac{2}{2.3.4}+\frac{2}{3.4.5}+$$ &c., to n terms, by subtraction.

$$\text{Whence } \frac{1}{4}-\frac{1}{2.(n+1)(n+2)}=\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+$$ &c., to n terms, by division.

And consequently $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+$ &c., continued to n terms $=\frac{1}{4}-\frac{1}{2.(n+1)(n+2)}=$ sum required.

14. Required the sum (s) of the series $\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+\frac{1}{32}-$ &c., continued ad infinitum.

$$\text{Let } x = \frac{1}{2} \text{ and } s = \frac{z}{1+x}.$$

$$\text{Then } \frac{z}{1+x} = (x - x^2 + x^3 - x^4 + x^5 \ \&c.)$$

$$\text{And } z = (1+x) \times (x - x^2 + x^3 - x^4 + x^5 \ \&c.)$$

Whence, by multiplication,

$$\begin{array}{l} x - x^2 + x^3 - x^4 + x^5 \ \&c. \\ 1 + x \\ \hline x - x^2 + x^3 - x^4 + x^5 - \&c. \\ \quad + x^2 - x^3 + x^4 - x^5 + \&c. \\ \hline \end{array}$$

The sum of which is $= x + 0 + 0 + 0 + 0 + \ \&c.$

Therefore $z = x$, and $x - x^2 + x^3 - x^4 + x^5 + \ \&c. = \frac{x}{1+x}$

Or $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} + \&c. = \frac{\frac{1}{2}}{1+\frac{1}{2}} = \frac{1}{3} =$ sum required.

15. Required the sum (S) of the series $\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16}$ $+ \ \&c.$, continued ad infinitum.

$$\text{Let } x = \frac{1}{2} \text{ and } s = \frac{z}{(1-x)^2}.$$

$$\text{Then } \frac{z}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + \ \&c.$$

$$\text{And } z = (1-x)^2 \times (x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + \ \&c.$$

Whence, by multiplication,

$$\begin{array}{l} x + 2x^2 + 3x^3 + 4x^4 + \ \&c. \\ 1 - 2x + x^2 \\ \hline x + 2x^2 + 3x^3 + 4x^4 \ \&c. \\ \quad - 2x^2 - 4x^3 - 6x^3 \ \&c. \\ \qquad\qquad + x^3 + 2x^4 \ \&c. \\ \hline \end{array}$$

The sum of which is $= x + 0 + 0 + 0 + \ \&c.$

Therefore $z=x$,

And $x+2x^2+3x^3+4x^4+5x^5$ &c.$=\dfrac{x}{(1-x)^2}$.

Or $\dfrac{1}{2}+\dfrac{2}{4}+\dfrac{3}{8}+\dfrac{4}{16}+\dfrac{5}{32}+\dfrac{6}{64}+$ &c. $=\dfrac{\frac{1}{2}}{(1-\frac{1}{2})^2}=2=$ sum of the infinite series required.

16. It is required to find the sum (s) of the series $\dfrac{1}{3}+\dfrac{4}{9}+\dfrac{9}{27}+\dfrac{16}{81}+\dfrac{25}{243}+$ &c., continued ad infinitum.

Let $x=\dfrac{1}{3}$ and $S=\dfrac{z}{(1-x)^3}$

Then $\dfrac{z}{(1-x)^3}=x+4x^2+9x^3+16x^4+25x^5+$ &c.

And $z=(1-x)^3\times(x+4x^2+9x^3+16x^4$ &c.$)=x+x^2$, as will be found by actual multiplication.

Therefore $x+x^2=z$,

And $x+4x^2+9x^3+16x^4+$ &c. $=\dfrac{x(1+x)}{(1-x)^3}$.

or

$\dfrac{1}{3}+\dfrac{4}{9}+\dfrac{9}{27}+\dfrac{16}{81}$ &c.$=\dfrac{\frac{1}{3}(1+\frac{1}{3})}{(1-\frac{1}{3})^3}=\dfrac{3}{2}=1\frac{1}{2}=$ sum required.

17. Required the sum (s) of the series $\dfrac{a}{m}+\dfrac{a+d}{mr}+\dfrac{a+2d}{mr^2}+\dfrac{a+3d}{mr^3}+$ &c. continued ad infinitum.

Let $x=\dfrac{1}{r}$, and $s=\dfrac{z}{m(1-x)^2}$

Then $\dfrac{z}{m(1-x)^2}=\dfrac{a}{m}+\dfrac{a+d}{mr}+\dfrac{a+2d}{mr^2}+\dfrac{a+3d}{mr^3}+$ &c.

Or $\frac{z}{(1-x)^2}=a+\frac{a+d}{r}+\frac{a+2d}{r^2}+\frac{a+3d}{r^3}+$ &c.

That is, $\frac{z}{(1-x)^2}=$

$a+(a+d)x+(a+2d)x^2+(a+3d)x^3+(a+4d)x^4+$ &c.

And $z=(1-x)^2\times\{a+(x+d)x+(a+2d)x^2+(a+3d)x^3+\&c.\}=(1-x)a+dx$,

as will appear by actually multiplying by $(1-x)^2$

Therefore $z=(1-x)a+dx$; and consequently $\frac{a}{m}+\frac{a+d}{mr}+\frac{a+2d}{mr^2}+\&c.=\frac{r}{m}\left\{\frac{a(r-1)+d}{(r-1)^2}\right\}=$ sum of the infinite series required.

EXAMPLES FOR PRACTICE.

1. Required the sum of 100 terms of the series 2, 5, 8, 11, 14, &c. Ans. 15050

2. Required the sum of 50 terms of the series $1+2^2+3^2+4^2+5^2+$ &c. Ans. 42925

3. It is required to find the sum of the series $1+3x+6x^2+10x^3+15x^4$ continued ad infinitum, &c., when x is a proper fraction, or less than 1. Ans. $\frac{1}{(1-x)^3}$

4. It is required to find the sum of the series $1+4x^2+10x^2+20x^3+35x^4+$ &c., continued ad infinitum, when x is a fraction less than 1. Ans. $\frac{1}{(1-x)^4}$

5. It is required to find the sum of the infinite series $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+$ &c. Ans. $\frac{1}{2}$

6. Required the sum of 40 terms of the series $(1\times2)+(3\times4)+(5\times6)+(7\times8)$ &c. Ans. 86884

7. Required the sum of n terms of the series $\frac{2x-1}{2x}$ $+\frac{2x-3}{2x}+\frac{2x-5}{2x}+\frac{2x-7}{2x}+$ &c. Ans. $n(\frac{2x-n}{2x})$

8. Required the sum of the infinite series $\frac{1}{1.2.3.4}+$ $\frac{1}{2.3.4.5}+\frac{1}{3.4.5.6}+\frac{1}{4.5.6.7}+$ &c. Ans. $\frac{1}{18}$

9. Required the sum of the series $\frac{1}{1}+\frac{1}{4}+\frac{1}{10}+\frac{1}{20}$ $+\frac{1}{35}+$ &c. continued ad infinitum. Ans. $\frac{3}{2}$, or $1\frac{1}{2}$

10. It is required to find the sum of n terms of the series $1+8x+27x^2+64x^3+125x^4+$ &c. Ans. $\frac{1+4x+x^2}{(1-x)^4}$

11. Required the sum of n terms of the series $\frac{1}{r}+\frac{2}{r^2}$ $+\frac{3}{r^3}+\frac{4}{r^4}+\frac{5}{r^5}+\frac{6}{r^6}+$ &c. Ans. $\frac{1}{(r-1)^2}-\frac{1}{r^n}\left\{\frac{nr+r-1}{(r-1)^2}\right\}$

12. Required the sum of the series $\frac{1}{2.6}+\frac{1}{4.8}+\frac{1}{6.10}$ $\frac{1}{8.12}+$ &c. . . . $+\frac{1}{2n(4+2n)}$. (c)

Ans. $\Sigma=\frac{3}{16}$, $S=\frac{5n+3n^2}{32+48n+16n^2}$

13. Required the sum of the series $\frac{1}{3.8}+\frac{1}{6.12}+\frac{1}{9.16}$ $+\frac{1}{12.20}+$ &c. . . . $+\frac{1}{3n(4+4n)}$.

Ans. $\Sigma=\frac{1}{12}$, $S=\frac{n}{12+12n}$

(c) The symbol Σ, made use of in these, and some of the following series, denotes the sum of an infinite number of terms, and S the sum of n terms.

14. Reqnired the sum of the series $\frac{6}{2.7}+\frac{6}{7.12}+\frac{6}{12.17}+\frac{6}{17.22}+\&c. \ldots +\frac{6}{(5n-3).(5n+2)}$.

Ans. $\Sigma=\frac{3}{5}$, $s=\frac{3n}{2+5n}$

15. Required the sum of the series $\frac{1}{3.6}-\frac{1}{6.8}+\frac{1}{9.10}-\frac{1}{12.12}+\&c. \ldots \pm\frac{1}{3n(4+2n)}$.

Ans. $\Sigma=\frac{1}{24}$, $s=\frac{n}{2(3+6n)}-\frac{n}{4(6+6n)}$

16. Required the sum of the series $\frac{2}{3.5}-\frac{3}{5.7}+\frac{4}{7.9}-\frac{5}{9.11}+\&c. \ldots \pm\frac{1+n}{(1+2n).(3+2n)}$.

Ans. $\Sigma=\frac{1}{12}$, $s=\frac{1}{12}-\frac{1}{4(3+4n)}$

17. Required the sum of the series $\frac{5}{1.2.3}+\frac{6}{2.3.4}+\frac{7}{3.4.5}+\frac{8}{4.5.6}+\&c. \ldots +\frac{4+n}{n(1+n).(2+n)}$.

Ans. $\Sigma=\frac{3}{2}$, $s=\frac{3}{2}-\frac{2}{1+n}+\frac{1}{2+n}$(*d*)

(*d*) The series here treated of are such as are usually called algebraical; which, of course, embrace only a small part of the whole doctrine. Those, therefore, who may wish for farther information on this abstruse, but highly curious subject, are referred to the *Miscellanea Analytica* of DEMOIVRE, STIRLING'S *Method. Differ.* JAMES BERNOULLI *de Seri. Infin.*, SIMPSON'S *Math. Dissert.*, WARING'S *Medit. Analyt.*, CLARKE'S translation

OF LOGARITHMS.

(M) LOGARITHMS are a set of numbers that have been computed and formed into tables, for the purpose of facilitating arithmetical calculations; being so contrived, that the addition and subtraction of them answer to the multiplication and division of the natural numbers with which they are made to correspond. (*e*)

Or, when taken in a similar, but more general sense, logarithms may be considered as the indices, or exponents of the powers to which a given, or invariable,

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of *Lorgna's Series*, the various works of EULER, and LACROIX *Traité du Calcul. Diff. et Int.*, where they will find nearly all the materials that have been hitherto collected respecting this branch of analysis.

(*e*) This mode of computation, which is one of the happiest and most useful discoveries, of modern times, is due to LORD NAPIER, Baron of Merchiston, in Scotland, who first published a table of these numbers, in the year 1614, under the title of *Canon Mirificum Logarithmorum;* which performance was eagerly received by the learned throughout Europe, whose efforts were immediately directed to the improvement and extension of the new calculus, that had so unexpectedly presented itself.

Mr. HENRY BRIGGS, in particular, who was, at that time, professor of Geometry in Gresham College, greatly contributed to the advancement of this doctrine, not only by the very advantageous alteration which he first introduced into the system of these numbers, by making one the logarithm of 10, instead of 2.3025852 . . ., as has been done by NAPIER; but also by the publication, in 1624, and 1633, of his two great works, the *Arithmetica Logarithmica*, and the *Trigonometria Britannica*, both of which were formed upon the principle above mentioned; as are, likewise, all our common logarithmic tables, at present in use.

See, for farther details on this part of the subject, the Introduction to my *Treatise of Plane and Spherical Trigonometry*, 8vo. 2d Edit. 1813; and for the construction and use of the tables, consult those of SHERWIN, HUTTON, TAYLOR, CALLET, and BORDA, where every necessary information, of this kind may be readily obtained.

number must be raised, in order to produce all the common, or natural numbers. Thus, if

$$a^x=y,\ a^{x'}=y',\ a^{x''}=y'',\ \&c.$$

then will the indices x, x', x'', &c. of the several powers of a, be the logarithms of the number, y, y', y'', &c. in the scale, or system of which a is the base.

So that, from either of these formulæ, it appears, that the logarithm of any number, taken separately, is the index of that power of some other number, which, when involved in the usual way, is equal to the given number.

And since the base a, in the above expressions, can be assumed of any value, greater or less than 1, it is plain that there may be an endless variety of systems of logarithms, answering to the same natural numbers.

It is, likewise, farther evident, from the first of these equations, that when $y=1$, x will be $=0$, whatever may be the value of a; and consequently the logarithm of 1 is always 0, in every system of logarithms.

And if $x=1$, it is manifest, from the same equation, that the base a will be $=y$; which base is, therefore, the number whose proper logarithm, in the system to which it belongs is 1.

Also, because $a^x=y$, and $a^{x'}=y'$, it follows, from the multiplication of powers, that $a^x\times a^{x'}$ or, $a^{x+x'}=yy'$; and consequently, by the definition of logarithms, given above, $x+x'=\log.\ yy'$ or

$$\log.\ yy'=\log.\ y+\log.\ y'.$$

And, for a like reason, if any number of the equations $a^x=y$, $a^{x'}=y'$, $a^{x''}=y''$, &c. be multiplied together, we shall have $a^{x+x'+x''\&c.}=yy'y''\&c.$; and, consequently $x+x'+x''\&c.=\log.\ yy'y''\ \&c.$; or

$$\log.\ yy'y''\ \&c.=\log.\ y+\log.\ y'+\log.\ y''\ \&c.$$

From which it is evident, that the logarithm of the product of any number of factors is equal to the sum of the logarithms of those factors.

Hence, if all the factors of a given number, in any case of this kind, be supposed equal to each other, and the sum of them be denoted by m, the preceding property will then become.

$$\log.\ y^m = m \log.\ y.$$

From which it appears, that the logarithm of the mth power of any number is equal to m times the logarithm of that number.

In like manner, if the equation $a^x = y$ be divided by $ax' = y'$, we shall have, from the nature of powers, as before $\frac{a^x}{a^{x'}}$, or $a^{x-x'} = \frac{y}{y'}$; and by the definition of logarithms, laid down, in the first part of this article, $x - x' = \log.\ \frac{y}{y'}$, or $\log.\ \frac{y}{y'} = \log.\ y - \log.\ y'$.

Hence the logarithm of a fraction, or of the quotient arising from dividing one number by another, is equal to the logarithm of the numerator *minus* the logarithm of the denominator, or to the logarithm of the dividend *minus* that of the divisor.

And if each member of the common equation $a^x = y$ be raised to the fractional power denoted by $\frac{m}{n}$, we shall have, in that case, $a^{\frac{m}{n}x} = y^{\frac{m}{n}}$;

And, consequently, by taking the logarithms of these quantities, as before $\frac{m}{n}x = \log.\ y^{\frac{m}{n}}$, or

$$\log.\ y^{\frac{m}{n}} = \frac{m}{n} \log.\ y$$

Where it appears, that the logarithm of a mixed root, or power, of any number, is found by multiplying the logarithm of the given number by the numerator of the index of that power, and dividing the result by the denominator.

And if the numerator m, of the fractional index, be,

in this case, taken equal to 1, the above formula will then become

$$\log. y^{\frac{1}{n}} = \frac{1}{n}\log. y.$$

From which it follows, that the logarithm of the nth root of any number is equal to the nth part of the logarithm of that number.

Hence, besides the use of logarithms in abridging the operations of multiplication and division, they are equally applicable to the raising of powers, and extracting of roots; which are performed by simply multiplying the given logarithm by the index of the power, or dividing it by the number denoting the root.

But, although the properties here mentioned are common to every system of logarithms, it was necessary, for practical purposes, to select some one of them from the rest, and to adapt the logarithms of all the natural numbers to that particular scale.

And as 10 is the base of our present system of arithmetic, the same number has accordingly been chosen for the base of the logarithmic system, now generally used.

So that, according to this scale, which is that of the common logarithmic tables, the numbers

$$\ldots\ 10^{-4},\ 10^{-3},\ 10^{-2},\ 10^{-1},\ 10^{0},\ 10^{1}, 10^{2},\ 10^{3},\ 10^{4}, \&c.$$

$$\text{Or}\ \ldots \frac{1}{10000}, \frac{1}{1000}, \frac{1}{100}, \frac{1}{10}, 1, 10, 100, 1000, 10000, \&c.$$

have for their logarithms

$$\ldots -4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4,\ \&c.$$

Which are evidently a set of numbers in arithmetical progression, answering to another set in geometrical progression; as is the case in every system of logarithms. (*f*)

And therefore, since the common, or tabular, loga-

(*f*) A detailed account of the theory and construction of logarithms, may be seen in Vol. II. of my *Treatise on Algebra.*

rithm of any number (n) is the index of that power of 10, which, when involved, is equal to the given number, it is plain, from the following equation,

$$10^x = n, \text{ or } 10^{-x} = \frac{1}{n},$$

that the logarithms of all the intermediate numbers, in the above series, may be assigned by approximation, and made to occupy their proper places in the general scale.

It is also evident, that the logarithms of 1, 10, 100, 1000, *&c.* being 0, 1, 2, 3, *&c.* respectively, the logarithm of any number, falling between 0 and 1, will be 0 and some decimal parts; that of a number, between 10 and 100, 1 and some decimal parts; of a number between 100 and 1000, 2 and some decimal parts; and so on, for other numbers of this kind.

In like manner, the logarithms of $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, *&c.* or of their equals .1, .01, .001, *&c.* in the descending part of the scale, being -1, -2, -3, *&c.* the logarithm of any number, falling between 0 and 1, will be -1 and some positive decimal parts; that of a number between 1 and .01, -2 and some positive decimal parts; of a number between .01 and .001, -3 and some positive decimal parts; *&c.*

Hence also, as the multiplying or dividing of any number by 10, 100, 1000, *&c.* is performed by barely increasing or diminishing the integral part of its logarithm by 1, 2, 3, *&c.* it is obvious that all numbers, which consist of the same figures, whether they be integral, fractional, or mixed, will have, for the decimal part of their logarithms, the same positive quantity.

So that, in this system, the integral part of any logarithm, which is usually called its index, or characteristic, is always less by 1 than the number of integers which the natural number consists of; and for decimals,

it is the number which denotes the distance of the first significant figure from the place of units.

Thus, according to the logarithmic tables in common use, we have

Numbers.	*Logarithms.*
1.36820	0.1361496
20.0500	1.3021144
335.260	2.5253817
.46521	$\bar{1}.6676490$
.06154	$\bar{2}.7891575$
&c.	*&c.*

Where the sign — is put over the index, instead of before it, when that part of the logarithm is negative, in order to distinguish it from the decimal part, which is always to be considered as +, or affirmative.

Also, agreeably to what has been before observed, the logarithm of 38540 being 4.5859117, the logarithms of any other numbers, consisting of the same figures, will be as follows:

Numbers.	*Logarithms.*
3854	3.5859117
385.4	2.5859117
38.54	1.5859117
3.854	0.5859117
.3854	$\bar{1}.5859117$
.03854	$\bar{2}.5859117$
.003854	$\bar{3}.5859117$

Which logarithms, in this case, as well as in all others of a similar kind, whether the number contains ciphers or not, differ only in their indices, the decimal or positive part, being the same in them all. (*g*)

(*g*) The great advantages attending the common, or BRIGGEAN system of logarithms above all others, arise chiefly from

And as the indices, or integral parts, of the logarithms of any numbers, whatever, in this system, can always be thus readily found, from the simple consideration of the rule above mentioned, they are generally omitted in the tables, being left to be supplied by the operator, as occasion requires.

It may here, also, be farther added, that, when the logarithm of a given number, in any particular system, is known, it will be easy to find the logarithm of the same number in any other system, by means of the following equations,

$ax=n$, and $e^{x'}=n$; or log. $n=x$, and $l.\ n=x'$.

Where log. denotes the logarithm of n, in the system of which a is the base, and l. its logarithm in the system of which e is the base.

For, since $a^x=e^{x'}$, or $a^{\frac{x}{x'}}=e$, and $e^{\frac{x'}{x}}=a$, we shall have, for the base a, $\frac{x}{x'}=\log.\ e$, or $x=x'\log.\ e$;

and for the base e, $\frac{x'}{x}=\mathrm{l}.\ a$, or $x'=x\mathrm{l}.\ a$.

Whence, by substitution, from the former equations,

$$\log.\ n=\mathrm{l}.\ n\times\log.\ e;\ \text{or}\ \log.\ n=\mathrm{l}.\ n\times\frac{1}{\mathrm{l}.a}.$$

Where the multiplier log. e, or its equal $\frac{1}{\mathrm{l}.a}$, ex-

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the readiness with which we can always find the characteristic or integral part of any logarithm from the bare inspection of the natural number to which it belongs, and the circumstance, that multiplying or dividing any number by 10, 100, 1000, *&c.*, only influences the characteristic of its logarithm, without affecting the decimal part. Thus, for instance, if i be made to denote the index, or integral part of the logarithm of any number N, and d its decimal part, we shall have

$$\text{Log. } N=i+d;\ \log.\ 10^m N=(i+m)+d;\ \log\ \frac{N}{10^m}=(i-m)+d$$

Where it is plain that the decimal part of the logarithm, in each of these cases, remains the same.

presses the constant relation which the logarithms of n have to each other in the systems to which they belong.

But the only system of these numbers, deserving of notice, except that above described, is the one that furnishes what have been usually called hyperbolic or NEPERIAN logarithms, the base e of which is 2.718281 828459. . . .

Hence, in comparing these with the common or tabular logarithms, we shall have, by putting a in the latter of the above formulæ $=10$, the expression

$$\log.\ n = \mathrm{l.}\ n \times \frac{1}{l.10}, \text{ or l. } n = \log.\ n \times \mathrm{l.}\ 10,$$

Where log. in this case, denotes the common tabular logarithm of the number n, and l. its hyperbolic logarithm; the constant factor, or multiplier, $\frac{1}{\mathrm{l.}10}$, which is

$$\frac{1}{2.3025850929}, \text{ or its equal } .4342944819,$$

being what is usually called the *modulus* of the common system of logarithms. (h)

PROBLEM I.

To compute the logarithm of any of the natural numbers 1, 2, 3, 4, 5, *&c.*

(h) It may here be remarked, that although the common logarithms have superseded the use of hyperbolic or NAPIERIAN logarithms, in all the ordinary operations to which these numbers are generally applied, yet the latter are not without some advantages peculiar to themselves; being of frequent occurrence in the application of the Fluxionary Calculus, to many analytical and physical problems, where they are required for the finding of certain fluents, which could not be so readily determined without their assistance; on which account, great pains have been taken to calculate tables of hyperbolic logarithms, to a considerable extent, chiefly for this purpose. Mr. BARLOW, in a *Collection of Mathematical Tables* lately published, has given them for the first 10,000 numbers.

RULE I.

1. Take the geometrical series, 1, 10, 100, 1000, 10000, *&c.*, and apply to it the arithmetical series, 0, 1, 2, 3, 4, *&c.*, as logarithms.

2. Find a geometric mean between 1 and 10, 10 and 100, or any other two adjacent terms of the series, betwixt which the number proposed lies.

3. Also, between the mean, thus found, and the nearest extreme, find another geometrical mean, in the same manner; and so on, till you are arrived within the proposed limit of the number whose logarithm is sought.

4. Find, likewise, as many arithmetical means between the corresponding terms of the other series, 0, 1, 2, 3, 4, *&c.*, in the same order as the geometrical means were found, and the last of these will be the logarithm answering to the number required.

EXAMPLES.

1. Let it be required to find the logarithm of 9.

Here the proposed number lies between 1 and 10.

First, then, the log. of 10 is 1, and the log. of 1 is 0.

Therefore $\sqrt{(10\times 1)}=\sqrt{10}=3.1622777$ is the geometrical mean;

And $\frac{1}{2}(1+0)=\frac{1}{2}=.5$ is the arithmetical mean;

Hence the log. of 3.1622777 is .5.

Secondly, the log. of 10 is 1, and the log. of 3.162 2777 is .5.

Therefore $\sqrt{(10\times 3.1622777)}=5.6234132$ is the geometrical mean;

And $\frac{1}{2}(1+.5)=.75$ is the arithmetical mean;

Hence the log. of 5.6234132 is .75.

Thirdly, the log. of 10 is 1, and the log. of 5.6234132 is .75;

Therefore $\sqrt{(10\times 5.6234132)}=7.4989422$ is the geometrical mean;

And $\frac{1}{2}(1\times .75)=.875$ is the arithmetical mean;

Hence the log. of 7.4989422 is .875.

Fourthly, the log. of 10 is 1, and the log. of 7.4989422 is .875;

Therefore $\sqrt{(10 \times 7.4989422)} = 8.6596431$ is the geometrical mean,

And $\frac{1}{2}(1 + .875) = .9375$ is the arithmetical mean;

Hence the log. of 8.6596431 is .9375.

Fifthly, the log. of 10 is 1, and the log. of 8.6596431 is .9375.

Therefore $\sqrt{(10 \times 8.6596431)} = 9.3057204$ is the geometrical mean,

And $\frac{1}{2}(1 + .9375) = 96875$ is the arithmetical mean;

Hence the log. of 9.3057204 is .96875.

Sixthly, the log. of 8.6596431 is .9375, and the log. of 9.3057204 is .96875;

Therefore $\sqrt{(8.6596431 \times 9.3057204)} = 8.9768713$ is the geometrical mean,

And $\frac{1}{2}(.9375 + .96875) = .953125$ is the arithmetical mean;

Hence the log. of 8.9768713. is .953125.

And, by proceeding in this manner, it will be found, after 25 extractions, that the logarithm of 8.9999998 is .9542425; which may be taken for the logarithm of 9, as it differs from it so little, that it may be considered as sufficiently exact for all practical purposes.

And in this manner were the logarithms of all the prime numbers at first computed.

RULE II.

When the logarithm of any number (n) is known, the logarithm of the next greater number may be found from the following series, by calculating a sufficient

number of its terms, and then adding the given logarithm to their sum.

$$\text{Log.}(n+1)=\log.n+2\text{M}\left\{\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\frac{1}{7(2n+1)^7}+\frac{1}{9(2n+1)^9}+\frac{1}{11(2n+1)^{11}}+\&c.\right\}$$

or

$$\text{Log.}(n+1)=\log.n+\left\{\frac{2\text{M}}{2n+1}+\frac{\text{A}}{3(2n+1)^2}+\frac{3\text{B}}{5(2n+1)^2}+\frac{5\text{C}}{7(2n+1)^2}+\frac{7\text{D}}{9(2n+1)^2}+\frac{9\text{E}}{11(2n+1)^2}+\&c.\right\}$$

Where A, B, C, *&c.*, represent the terms immediately preceding those in which they are first used; the modulus $\text{M}=.4342944819 \ldots$; and consequently, $2\text{M}=.8685889638 \ldots$ (*i*)

EXAMPLES.

1. Let it be required to find the common logarithm of the number 2.

Here, because $n+1=2$, and consequently $n=1$ and $2n+1=3$, we shall have

$$\frac{2\text{M}}{2n+1}=\frac{.8685889638}{3}=.289529654 \text{ (A)}$$

$$\frac{\text{A}}{3(2n+1)^2}=\frac{.289529654}{3.3^2}=.010723321 \text{ (B)}$$

(*i*) It may here be remarked, that the difference between the logarithms of any two consecutive numbers, is so much the less as the numbers are greater; and consequently the series which comprises the latter part of the above expression, will in that case converge so much the faster. Thus

$\log.\ n$, and $\log.\ (n+1)$, or its equal $\log.\ n+\log.\ (1+\frac{1}{n})$

will, obviously, differ but little from each other when n is a large number.

$$\frac{3B}{5(2n+1)^2}=\frac{3\times.010723321}{5.3^2}=.000714888\ (C)$$

$$\frac{5C}{7(2n+1)^2}=\frac{5\times.000714888}{7.3^2}=.000056737\ (D)$$

$$\frac{7D}{9(2n+1)^2}=\frac{7\times.000056737}{9.3^2}=.000004903\ (E)$$

$$\frac{9E}{11(2n+1)^2}=\frac{9\times.000004903}{11.3^2}=.000000446\ (F)$$

$$\frac{11F}{13(2n+1)^2}=\frac{11\times.000000446}{13.3^2}=.000000042\ (G)$$

$$\frac{13G}{15(2n+1)^2}=\frac{13\times.000000042}{15.3^2}=.000000004\ (H)$$

Sum of 8 terms . . .	.301029995
Add log. of 1 . . .	.000000000
Log. of 2	.301029995

Which logarithm is true to the last figure inclusively.

2. Let it be required to compute the logarithm of the number 3.

Here, since $n+1=3$, and consequently $n=2$, and $2n+1=5$, we shall have

$$\frac{2M}{2n+1}=\frac{.868588964}{5}\ .\ .=.173717793\ (A)$$

$$\frac{A}{3(2n+1)^2}=\frac{.173717793}{3.5^2}\ .\ .=.002316237\ (B)$$

$$\frac{3B}{5(2n+1)^2}=\frac{3\times.002316237}{5.5^2}\ .\ .=.000055590\ (C)$$

$$\frac{5\text{C}}{7(2n+1)^2}=\frac{5\times.000055590}{7.5^2}=.000001588\ (\text{D})$$

$$\frac{7\text{D}}{9(2n+1)^2}=\frac{7\times.000001588}{9.5^2}=.000000050\ (\text{E})$$

$$\frac{9\text{E}}{11(2n+1)^2}=\frac{9\times.000000050}{11.5^2}=.000000002\ (\text{F})$$

Sum of 6 terms . . .	.176091260
Log. of 2	.301029995
Log. of 3	.477121255

Which logarithm is also correct to the nearest unit in the last figure.

And in the same way we may proceed to find the logarithm of any prime number.

Also, because the sum of the logarithms of any two numbers gives the logarithm of their product, and the difference of the logarithms the logarithm of their quotient, &c.; we may readily compute, from the above two logarithms, and the logarithm of 10, which is 1, a great number of other logarithms, as in the following examples:

3. Because $2\times2=4$, therefore log. 2 .301029995
mult. by 2   2

gives log. 4 .602059990

4. Because $2\times3=6$, therefore to log. 2 .301029995
add log. 3 .477121255

gives log. 6 .778151250

5. Because $2^3=8$, therefore	log. 2	.301029995
	mult. by 3	3
	gives log. 8	.903089985
6. Because $3^2=9$, therefore	log. 3	.477121255
	mult. by 2	2
	gives log. 9	.954242510
7. Because $\frac{10}{2}=5$, therefore	from log. 10	1.000000000
	take log 2	.301029995
	gives log. 5	.698970005
8. Because $3\times4=12$, therefore	to log. 3	.477121255
	add log. 4	.602059991
	gives log. 12	1.079181246

And thus, by computing, according to the general formula, the logarithms of the next succeeding prime numbers 7, 11, 13, 17, 19, 23, *&c.* we can find, by means of the simple rules, before laid down for multiplication, division and the raising of powers, as many other logarithms as we please, or may speedily examine any logarithm in the table.

MULTIPLICATION BY LOGARITHMS.

Take out the logarithms of the factors from the table, and add them together; then the natural number answering to the sum will be the product required.

Observing, in the addition, that what is to be carried

from the decimal part of the logarithms is always affirmative, and must, therefore, be added to the indices, or integral parts, after the manner of positive and negative quantities in algebra.

Which method will be found much more convenient, to those who possess a slight knowledge of this science, than that of using the arithmetical complements.

EXAMPLES.

1. Multiply 37.153 by 4.086, by logarithms.

Nos.		*Logs.*
37.153	. . . . .	1.5699939
4.086	. . . . .	0.6112984
Prod. 151.8071	.	2.1812923

2. Multiply 112.246 by 13.958, by logarithms.

Nos.		*Logs.*
112.246	. . . . .	2.0491709
13.958	. . . . .	1.1448232
Prod. 1563.128	.	3.1939941

3. Multiply 46.7512 by .3275, by logarithms.

Nos.		*Logs.*
46.7512	. . . . .	1.6697928
.3275	:	$\bar{1}$.5152113
Prod. 15.31102	.	1,1850041

Here, the +1, that is to be carried from the decimals, cancels the —1, and consequently there remains 1 in the upper line to be set down.

4. Multiply .37816 by .04782, by logarithms.

Nos.	*Logs.*
.37816	$\bar{1}.5776756$
.04782	$\bar{2}.6796096$
Prod. .0180836 .	$\bar{2}.2572852$

Here the + 1 that is to be carried from the decimals, destroys the — 1, in the upper line, as before, and there remains the — 2 to be set down.

5. Multiply 3.768,2.053 and .007693, together.

Nos.	*Logs.*
7.768	0.5761109
2.053	0.3123889
.007693	$\bar{3}.8860997$
Prod. .059511 .	$\bar{2}.7745995$

Here the + 1, that is to be carried from the decimals, when added to — 3, makes — 2, to be set down.

6. Multiply 3.586, 2.1046, .8372 and .0294, together.

Nos.	*Logs.*
3.586	0.554610
2.1046	0.323170
.8372	$\bar{1}.922829$
.0294	$\bar{2}.468347$
Prod..1857618 . .	$\bar{1}.268956$

Here the + 2, that is to be carried, cancels the — 2 and there remains the — 1 to be set down.

7. Multiply 23.14 by 5.062 by logarithms.
Ans. 117.1347

8. Multiply 4.0763 by 9.8432, by logarithms.
Ans. 40.12383

9. Multiply 498.256 by 41.2467, by logarithms.
Ans. 20551.41

10. Multiply 4.26747 by .012345, by logarithms.
Ans. .0497116

11. Multiply 3.12567, .02868 and .12379, together, by logarithms. Ans. .01109705

12. Multiply 2876.9, .10674, .098762 and 0031598, by logarithms. Ans. .0958299

DIVISION BY LOGARITHMS.

From the logarithm of the dividend, as found in the tables, subtract the logarithm of the divisor, and the natural number, answering to the remainder, will be the quotient required.

Observing, if the subtraction cannot be made in the usual way, to add, as in the former rule, the 1 that is to be carried from the decimal part, when it occurs, to the index of the logarithm of the divisor, and then this result, with its sign changed, to the remaining index, for the index of the logarithm of the quotient.

EXAMPLES.

1. Divide 4768.2 by 36.954, by logarithms.

Nos.	*Logs.*
4768.2	3.6783545
36.954	1.5676615
Quot. 129.032 . .	2.1106930

2. Divide 21.754 by 2.4678, by logarithms.

Nos.		*Logs.*
21.754	. . . . .	1.3375391
2.4678	. . . . .	0.3923100
Quot. 81518	. .	0.9452291

3. Divide 4.6257 by .17608, by logarithms.

Nos.		*Logs.*
4.6257	. . . . .	0.6651725
.17608	. . . . .	$\bar{1}$.2457100
Quot. 26.2741	.	1.4194625

Here -1, in the lower index, is changed into $+1$, which is then taken for the index of the result.

4. Divide .27684 by 5.1576, by logarithms.

Nos.		*Logs.*
.27684	. . . . .	$\bar{1}$.4422288
5.1576	. . . . .	0.7124477
Quot. .0536761	.	$\bar{2}$.7297811

Here the 1 that is to be carried from the decimals, is taken as -1, and then added to -1, in the upper index, which gives -2 for the index of the result.

5. Divide 6.9875 by .075789, by logarithms.

Nos.		*Logs.*
6.9875	. . . . .	0.8443218
.075789	. . . . .	$\bar{2}$.8796062
Quot. 92.1967	. .	1.9647156

Here the 1, that is to be carried from the decimals, is added to -2, which makes -1, and this put down, with its sign changed, is $+1$.

6. Divide .19876 by .0012345, by logarithms.

Nos.	*Logs.*
.19876	$\bar{1}.2983290$
.0012345 . . .	$\bar{3}.0914911$
Quot. 161.0051 .	2.2068379

Here -3, in the lower index, is changed into $+3$, and this added to -1, the other index, gives $+3-1$ or 2.

7. Divide 125 by 1728, by logarithms.
Ans. .0723379

8. Divide 1728.95 by 1.10678, by logarithms.
Ans. .1562.144

9. Divide 10.23674 by 4.96523, by logarithms.
Ans. 2.061685

10. Divide 19956.7 by .048235, by logarithms.
Ans. 413739

11. Divide .067859 by 1234.59, by logarithms.
Ans. 0000549648

THE RULE OF THREE,

OR PROPORTION, BY LOGARITHMS.

For any single proportion, add the logarithms of the second and third terms together, and subtract the logarithm of the first from their sum, according to the foregoing rules; then the natural number answering to the result will be the fourth term required.

But if the proportion be compound, add together the logarithms of all the terms that are to be multiplied, and from the result take the sum of the logarithms of the other terms, and the remainder will be the logarithm of the term sought.

Or, the same may be performed more conveniently thus,

Find the complement of the logarithm of the first term of the proportion, or what it wants of 10, by beginning at the left hand, and taking each of its figures from 9, except the last significant figure, on the right, which must be taken from 10; then add this result and the logarithms of the other two terms together, and the sum, abating 10 in the index, will be the logarithm of the fourth term, as before.

And, if two or more logarithms are to be subtracted, as in the latter part of the above rule, add their complements and the logarithms of the terms to be multiplied together, and the result, abating as many 10's in the index as there are logarithms to be subtracted, will be the logarithm of the term required; observing, when the index of the logarithm, whose complement is to be taken, is negative, to add it, as if it were affirmative, to 9; and then take the rest of the figures from 9, as before.

EXAMPLES.

1. Find a fourth proportional to 37.125, 14.768 and 135.279, by logarithms.

Log. of 37.125 . . .	1.5696665
Complement . . .	8.4303335
Log. of 14.768 . . .	1.1693217
Log. of 135.279 . . .	2.1312304
Ans. 53.81099 . . .	1.7308856

3. Find a fourth proportional to .05764, .7186 and .34721, by logarithms.

Log. of .05764 . . .	$\bar{2}.7607240$
Complement . . .	11.2392760
Log. of .7186 . . .	$\bar{1}.8564872$
Log. of .34721 . . .	$\bar{1}.5405922$
Ans. 4.328681 . .	0.6363554

3. Find a third proportional to 12.796 and 3.24718, by logarithms.

Log. of 12.796 . . .	1.1070742
Complement . . .	8.8929258
Log. of 3.24718 . . .	0.5115064
Log. of 3.24718 . . .	0.5115064
Ans. .8240216 . .	$\bar{1}.9159386$

4. Find the interest of 279*l.* 5*s.* for 274 days, at $4\frac{1}{2}$ per cent. per annum, by logarithms.

Comp. log. of 100 . .	8.0000000
Comp. log. of 365 . .	7.4377071
Log. of 279.25 . . .	2.4459932
Log. of 274	2.4377506
Log. of 4.5	0.6532125
Ans. 9.433296 . .	0.9746634

5. Find a fourth proportional to 12.678, 14.065 and 100.979, by logarithms. Ans. 112.0263

6. Find a fourth proportional to 1.9864, .4678 and 50.4567, by logarithms. Ans. 11.88262

7. Find a fourth proportional to .09658, .24958 and .008967, by logarithms. Ans. .02317234

8. Find a third proportional to .498621 and 2.9587 and a third proportional to 12.796 and 3.24718 by logarithms. Ans. 17.55623 and .8240216

INVOLUTION,

OR THE RAISING OF POWERS BY LOGARITHMS.

Take out the logarithm of the given number from the tables, and multiply it by the index of the proposed power; then the natural number, answering to the result, will be the power required.

Observing, if the index of the logarithm be negative, that this part of the product will be negative; but as what is to be carried from the decimal part will be affirmative, the index of the result must be taken accordingly.

EXAMPLES.

1. Find the square of 2.7568, by logarithms.

Log. of 2.7568 . . .	0.4402477
	2
Square 7.599946 . .	0.8804954

2. Find the cube of 7.0851, by logarithms.

Log. of 7.0851 . . .	0.8503399
	3
Cube 355.6475 . . .	2.5510197

Therefore 355.6475 the answer.

3. Find the fifth power of .87451, by logarithms.

$$\begin{array}{lr} \text{Log. } .87451 \quad \ldots & \bar{1}.9417648 \\ & 5 \\ \hline \text{Fifth power } .5114695 \quad . & \bar{1}.7088240 \\ \hline \end{array}$$

Where 5 times the negative index $\bar{1}$, being -5, and $+4$ to carry, the index of the power is $\bar{1}$.

4. Find the 365th power of 1.0045, by logarithms.

$$\begin{array}{llr} \text{Log. } 1.0045 & \ldots & 0.0019499 \\ & & 365 \\ \hline & & 97495 \\ & & 116994 \\ & & 58497 \\ \hline \text{Power } 5.148888 & \text{Log.} & 0.7117135 \\ \hline \end{array}$$

Therefore 5.148888 = power required.

5. Required the square of 6.05987, by logarithms.
Ans. 36.72203

6. Required the cube of .176546, by logarithms.
Ans. .005502674

7. Required the 4th power of .076543 by logarithms.
Ans. .0000343259

8. Required the 5th power of 2.97643 by logarithms.
Ans. 233.6031

9. Required the 6th power of 21.0576 by logarithms.
Ans. 87187340

10. Required the 7th power of 1.09684, by logarithms.
Ans. 1.909864

EVOLUTION,

OR THE EXTRACTION OF ROOTS, BY LOGARITHMS.

Take out the logarithm of the given number from the table, and divide it by 2 for the square root, 3 for the cube root, &c., and the natural number answering to the result will be the root required.

But if it be a compound root, or one that consists both of a root and a power, multiply the logarithm of the given number by the numerator of the index, and divide the product by the denominator, for the logarithm of the root sought.

Observing, in either case, when the index of the logarithm is negative, and cannot be divided without a remainder, to increase it by such a number as will render it exactly divisible; and then carry the units borrowed, as so many tens, to the first figure of the decimal part, and divide the whole accordingly.

EXAMPLES.

1. Find the square root of 27.465, by logarithms.

Log. of 27.465 . .	2)1.4387796
Root 5.2407 . . .	.7193898

2. Find the cube root of 35.6415, by logarithms.

Log. of 35.6415 . .	3)1.5519560
Root 3.29093 . . .	.5173186

3. Find the 5th root of 7.0825, by logarithms.

Log. of 7.0825 . .	5)0.8501866
Root 1.479235 . .	.1700373

4. Find the 365th root of 1.045, by logarithms.

Log. of 1.045 . .	365)0.0191163
Root 1.000121 . .	0.0000524

5. Find the value of $(.001234)^{\frac{2}{3}}$ by logarithms.

Log. of 001234 . . .	$\bar{3}.0913152$
	2
	3)6.1826304
Ans. 00115047 . . .	$\bar{2}.0608768$

Here, the divisor 3 being contained exactly twice in the negative index -6, the index of the quotient, to be put down, will be -2.

Find the value of $(.024554)^{\frac{3}{2}}$, by logarithms.

Log. of .024554 . .	$\bar{2}.3901223$
	3
	2)$\bar{5}.1703669$
Ans. .00384754 . . .	$\bar{3}.5851834$

Here 2 not being contained exactly in -5, 1 is added to it, which gives -3 for the quotient; and the 1 that is borrowed being carried to the next figure, makes 11, which, divided by 2, gives .5851834 for the decimal part of the logarithm.

7. Required the square root of 365.5674, by logarithms. Ans. 19.11981

8. Required the cube root of 2.987635, by logarithms. Ans. 1.440265

9. Required the 4th root of .967845, by logarithms.
Ans. .9918624

10. Required the 7th root of .098674, by logarithms.
Ans. .7183146

11. Required the value of $(\frac{21}{373})^{\frac{2}{3}}$, by logarithms.
Ans. .146895

12. Required the value of $(\frac{112}{1727})^{\frac{3}{5}}$, by logarithms.
Ans. .1937115

MISCELLANEOUS EXAMPLES IN LOGARITHMS.

1. Required the square root of $\frac{2}{123}$, by logarithms.
Ans. .1275153

2. Required the cube root of $\frac{1}{3.14159}$, by logarithms.
Ans. .6827842

3. Required the .07 power of .00563, by logarithms.
Ans. .6958821

4. Required the value of $\frac{(\frac{2}{3})^{\frac{1}{2}} \times (\frac{3}{4})^{\frac{1}{3}}}{17\frac{1}{3}}$, by logarithms.
Ans. .04279825

5. Required the value of $\frac{1}{7}\sqrt{\frac{5}{8}} \times .012\sqrt[3]{\frac{7}{11}}$, by logarithms.
Ans. .001165713

6. Required the value of $\frac{\frac{1}{9}\sqrt{\frac{11}{21}} \times .03\sqrt[3]{15\frac{1}{5}}}{7\frac{1}{3}\sqrt[3]{12\frac{1}{5}} \times .19\sqrt[4]{17\frac{1}{8}}}$, by logarithms.
Ans. .0009158638

7. Required the value of $\frac{127}{4}(\frac{\frac{5}{6}\sqrt{19} + \frac{4}{7}\sqrt[3]{35\frac{1}{3}}}{14\frac{7}{19} - \frac{15}{11}\sqrt{28\frac{2}{3}}})$, by logarithms.
Ans. 24.739

MISCELLANEOUS QUESTIONS.

1. A person being asked what o'clock it was, said it is between eight and nine, and the hour and minute hands are exactly together; what was the time?

Ans. 8h. 43 min. 38 $\frac{2}{11}$ sec.

2. A certain number, consisting of two places of figures, is equal to the difference of the squares of its digits, and if 36 be added to it the digits will be inverted; what is the number? Ans. 48

3. What two numbers are those, whose difference, sum, and product, are to each other as the numbers 2, 3, and 5, respectively? Ans. 2 and 10

4. A person, in a party at cards, betted three shillings to two upon every deal, and after twenty deals found he had gained five shillings; how many deals did he win? Ans. 13

5. A person wishing to enclose a piece of ground with palisades, found, if he set them a foot asunder, that he should have too few by 150, but if he set them a yard asunder he should have too many by 70; how many had he? Ans. 180

6. A cistern will be filled by two cocks, A and B, running together, in twelve hours, and by the cock A alone in twenty hours, in what time will it be filled by the cock B alone? Ans. 30 hours

7. A grocer bought a lot of tea at 10*s.* a lb., and a quantity of coffee at 2*s.* 6*d.* a lb.; which cost him altogether 31*l.* 5*s.*: but the state of the market having changed, he sold the tea at 8*s.* a lb., and the coffee at 4*s.* 6*d.* a lb., and gained upon the whole 5*l.*, how much of each did he buy?

Ans. 40 lbs. of tea, and 90 lbs. of coffee

8. What number is that, which, being severally added to 3, 19, and 51, shall make the results in geometrical progression? Ans. 13

9. It is required to find two geometrical mean proportionals between 3 and 24; and four geometrical means between 3 and 96.

Ans. 6 and 12; and 6, 12, 24, and 48

10. It is required to find six numbers in geometrical progression such, that their sum shall be 315, and the sum of the two extremes 165.

Ans. 5. 10, 20, 40, 80, and 160

11. It is required to find the length and breadth of a rectangular field, consisting of two acres of ground, that shall have the same perimeter as a square field consisting of four acres. Ans. 43.1906, and 7.4050 poles

12. After a certain number of men had been employed on a piece of work for 24 days, and had half finished it, 16 men more were set on, by which the remaining half was completed in 16 days: how many men were employed at first; and what was the whole expense, at 1*s.* 6*d.* a day per man? Ans. 32 the number of men; and the whole expense 115*l.* 4*s.*

13. It is required to find two numbers, such that if the square of the first be added to the second, the sum shall be 62, and if the squares of the second be added to the first, it shall be 176. Ans. 7 and 13

14. The fore wheel of a carriage makes six revolutions more than the hind wheel, in going 120 yards; but if the circumference of each wheel was increased by three feet, it would make only four revolutions more than the hind wheel in the same space; what is the circumference of each wheel? Ans. 12 and 15 feet

15. A person bought as many sheep as cost him 98*l.* 16*s.*; one-third of which he sold again at 40*s.* a piece, one-fourth at 36*s.*, and the rest at 34*s.* a piece; and found his gain upon the whole to be 10*l.* 14*s.*; what number of sheep had he? Ans. 60

16. A bankrupt owes A twice as much as he owes B, and C as much as he owes A and B together; now, out of 300*l.*, which is to be divided amongst them, what must each receive? Ans. A 100*l.*, B 50*l.*, and C 150*l.*

17. A sum of money is to be divided equally among a certain number of persons; now if there had been 3 claimants less, each would have had 150*l.* more, and if there had been 6 more, each would have had 120*l.* less; required the number of persons, and the sum divided.
Ans. 9 persons, sum 2700*l.*

18. From each of sixteen foreign pieces of gold, of the same denomination, a person filed a fifth of its value and then offered them all in payment at their nominal currency; but the fraud being detected, and the pieces weighed, they were found to be worth no more than 11*l.* 4*s.*; what was the original value of each piece?
Ans. 17*s.* 6*d.*

19. A composition of tin and copper, containing 100 cubic inches, was found to weigh 505 ounces; how many ounces of each did it contain, supposing the weight of a cubic inch of copper to be $5\frac{1}{4}$ ounces, and that of a cubic inch of tin $4\frac{1}{4}$ ounces.
Ans. 420 oz. of copper, and 85 oz. of tin.

20. A and B formed a joint stock in trade of 500*l.*, and cleared by the first bargain they made 160*l.*; out of which A's share came to 32*l.* more than that of B: what sum did each of them advance. Ans. A 300*l.*, and B 200*l.*

21. In how many different ways is it possible to pay 100*l.* with seven shilling pieces, and dollars of 4*s.* 6*d.* each? Ans. 21 different ways

22. The sum of two numbers is 2. and the sum of

their ninth powers is 32; required the numbers by a quadratic equation. Ans. $1 \pm \sqrt{\left\{\frac{2\sqrt{34}-11}{3}\right\}}$

23. It is required to find two numbers, such that their product shall be equal to the difference of their squares, and the sum of their squares equal to the difference of their cubes. Ans. $\frac{1}{2}\sqrt{5}$ and $\frac{1}{4}(5+\sqrt{5})$

24. The arithmetical mean of two numbers exceeds the geometrical mean by 13, and the geometrical mean exceeds the harmonical mean by 12; what are the numbers? Ans. 234 and 104

25. Given $xy(x^2+y^2)=3$, and $x^2y^2(x^4+y^4)=7$, to find the values of x and y.

Ans. $x=\frac{1}{2}(\sqrt{5}+1)$, $y=\frac{1}{2}(\sqrt{5}-1)$

26. Given $x+y+z=23$, $xy+xz+yz=167$, and $xyz=385$, to find x, y, and z. Ans. $x=5$, $y=7$, $z=11$

27. To find four numbers, x, y, z, and w, having the product of every three of them given; viz. $xyz=231$, $xyw=420$, $yzw=1540$, and $xzw=660$.

Ans. $x=3$, $y=7$, $z=11$, and $w=20$

28. Given $x+yz=384$, $y+xz=237$, and $z+xy=192$, to find the values of x, y, and z.

Ans. $x=10$, $y=17$, and $z=22$

29. Given $x^2+xy=108$, $y^2+yz=69$, and $z^2+xz=580$, to find the values of x, y, and z.

Ans. $x=9$, $y=3$, and $z=20$

30. Given $x^2+xy+y^2=5$, and $x^4+x^2y^2+y^4=11$, to find the values of x and y by a quadratic.

Ans. $x=\frac{2}{5}\sqrt{10}+\frac{1}{5}\sqrt{5}$, $y=\frac{2}{5}\sqrt{10}-\frac{1}{5}\sqrt{5}$

31. Given the equation $x^{4n}-2x^{3n}+x^n=a$, to find the value of x by a quadratic.

Ans. $x=\frac{1}{2}+\sqrt[n]{\left\{\frac{3}{4}\pm\sqrt{\left(a+\frac{3}{4}\right)}\right\}}$

32. It is required to find by what part of the population a people must increase annually, so that they may be doubled at the end of every century.

Ans. By *a* 144*th part nearly*

33. Required the least number of weights, and the weight to each, that will weigh any number of pounds from one pound to a hundred weight.

Ans. 1, 3, 9, 27, 81

34. A person bought as many ducks and geese together as cost him 28*s*., for the geese he paid 4*s*. 4*d*. a piece, and for the ducks 2*s*. 6*d*. a piece; what number of each had he? Ans. 3 geese and 6 ducks

35. It is required to find the least number, which being divided by 6, 5, 4, 3, and 2, shall leave the remainders 5, 4, 3, 2, and 1, respectively Ans. 59

36. Given the cycle of the sun 18, the golden number or cycle of the moon 8, and the Roman indiction 10, to find the year. Ans. 1717

37. Given $256x-87y=1$, to find the least possible values of x and y in whole numbers.

Ans. $x=52$, and $y=153$

38. It is required to find two different isosceles triangles, such that their perimeters and areas shall be both expressed by the same numbers.

Ans. Sides of the one 29, 29, 40; and of the other 37, 37, 24

39. It is required to find the sides of three right angled triangles, in whole numbers, such that their areas shall be all equal to each other.

Ans. 58, 40, 42; 74, 24, 70; 113, 15, 112

40. Given $x^{\frac{1}{x}}=1.2655$, to find a near approximate value of x. Ans. 3.82013

41. Given $x^y=5000$, and $y^x=3000$, to find the values of x and y. Ans. $x=4.691445$, and $y=5.510132$

42. Given $x^x+y^y=285$, and $y^x-x^y=14$, to find the values of x and y.

Ans. $x=4.016698$, and $y=2.825716$

43. To find two whole numbers such, that if unity be added to each of them, and also to their halves, the sums, in both cases, shall be squares. Ans. 48 and 1680

44. Required the two least nonquadrate numbers x and y, such that the sum of their squares, and the sum of their cubes, shall be both squares.

Ans. $x=364$ and $y=273$

45. It is required to find two whole numbers such, that their sum shall be a cube, and their product and quotient squares.

Ans. 25 and 100, or 100 and 900, *&c.*

46. It is required to find three biquadrate numbers such, that their sum shall be a square.

Ans. 12^4, 15^4, and 20^4

47. It is required to find three numbers in continued geometrical progression, such that their three differences shall be all squares. Ans. 567, 1008, and 1792

48. It is required to find three whole numbers such, that the sum or difference of any two of them shall be square numbers. Ans. 856350, 949986, and 993250

49. It is required to find two whole numbers such, that their sum shall be a square, and the sum of their squares a biquadrate.

Ans. 4565486027761 and 1061652293520

50. It is required to find four whole numbers such, that the difference of every two of them shall be a square number.

Ans. 1873432, 2288168, 2399057, and 6560657

51. It is required to find the sum of the series $\frac{1}{3}+\frac{2}{9}+\frac{3}{27}+\frac{4}{81}+$ &c. continued to infinity. Ans. $\frac{3}{4}$

52. It is required to find the sum of the infinite series $\frac{3}{4}-\frac{9}{16}+\frac{27}{64}-\frac{81}{256}+\frac{243}{1024}-$ &c. Ans. $\frac{3}{7}$

53. It is required to find the approximate value of the infinite series $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-$ &c. Ans. .822467

54. It is required to find the sum of the series $5+6+7+8+9+$ &c. continued to n terms. Ans. $\frac{n}{2}(n+9)$

55. It is required to find how many figures it would take to express the 25th term of the series $2^1+2^2+2^4+2^8+2^{16}+$ &c. Ans. 5050446 figures

56. It is required to find the sum of 100 terms of the series $(1\times 2)+(3\times 4)+(5\times 6)+(7\times 8)+(9\times 10)+$ &c. Ans. 343400

57. Required the sum of $1^2+2^2+3^2+4^2+5^2+$ &c. .. $+50^2$, which gives the number of shot in a square pile, the side of which is 50. Ans. 42925

58. Required the sum of 25 terms of the series $35+36\times 2+37\times 3+38\times 4+39\times 5$ &c., which gives the number of shot in a complete oblong pile, consisting of 25 tiers, the number of shot in the uppermost row being 35. Ans. 16575

APPENDIX.

OF THE APPLICATION OF ALGEBRA TO GEOMETRY.

In the preceding part of the present performance, I have considered Algebra as an independent science, and confined myself chiefly to the treating on such of its most useful rules and operations as could be brought within a moderate compass; but as the numerous applications, of which it is susceptible, ought not to be wholly overlooked, I shall here show, in compliance with the wishes of many respectable teachers, its use in the resolution of geometrical problems; referring the reader to my larger work on this subject, for what relates more immediately to the general doctrine of curves. (*k*)

For this purpose it may be observed, that when any proposition of the kind here mentioned is required to be resolved algebraically, it will be necessary, in the first place, to draw a figure that shall represent the several

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(*k*) The learner, before he can obtain a competent knowledge of the method of application above mentioned, must first make himself master of the principal propositions of Euclid, or of those contained in my *Elements of Geometry*; in the latter of which works he will find all the essential principles of the science comprised within a much shorter compass than in the former.

And in those cases where it may be requisite to extend this mode of application to trigonometry, mechanics, or any other branch of mathematics, a previous knowledge of the nature and principles of these subjects will be equally necessary.

parts or conditions, of the problem under consideration, and to regard it as the true one.

Then, having properly considered the nature of the question, the figure so formed, must, if necessary, be still farther prepared for solution, by producing, or drawing, such lines in it as may appear, by their connexion or relations to each other, to be most conducive to the end proposed.

This being done, let the unknown line, or lines, which it is judged will be the easiest to find, together with those that are known, be denoted by the common algebraic symbols, or letters; then, by means of the proper geometrical theorems, make out as many independent equations as there are unknown quantities employed; and the resolution of these, in the usual manner, will give the solution of the problem.

But as no general rules can be laid down for drawing the lines here mentioned, and selecting the properest quantities to substitute for, so as to bring out the most simple conclusions, the best means of obtaining experience in these matters will be to try the solution of the same problem in different ways; and then to apply that which succeeds the best to other cases of the same kind, when they afterwards occur.

The following directions, however, which are extracted, with some alterations, from NEWTON'S *Universal Arithmetic*, and SIMPSON'S *Algebra*, and *Select Exercises*, will often be found of considerable use to the learner, by showing him how to proceed in many cases of this kind, where he would otherwise be left to his own judgment.

1st. In preparing the figure in the manner above mentioned, by producing or drawing certain lines, let them be either parallel or perpendicular to some other lines in it, or be so drawn as to form similar triangles; and, if an angle be given, let the perpendicular be drawn oppo-

site to it, and so as to fall, if possible, from one end of a given line.

2d. In selecting the proper quantities to substitute for, let those be chosen, whether required or not, that are nearest to the known or given parts of the figure, and by means of which the next adjacent parts may be obtained by addition or subtraction only, without using surds.

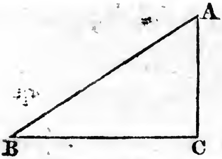
3d. When, in any problem, there are two lines, or quantities, alike related to other parts of the figure or problem, the best way is not to make use of either of them separately, but to substitute for their sum, difference, or rectangle, or the sum of their alternate quotients; or for some other line or lines in the figure, to which they have both the same relation.

4th. When the area, or the perimeter, of a figure is given, or such parts of it as have only a remote relation to the parts that are to be found, it will sometimes be of use to assume another figure similar to the proposed one, that shall have one of its sides equal to unity, or to some other known quantity; as the other parts of the figure, in such cases, may then be determined by the known proportions of their like sides, or parts; and thence the resulting equation required.

These being the most general observations that have hitherto been collected upon this subject, I shall now proceed to elucidate them by proper examples; leaving such farther remarks as may arise out of the mode of proceeding here used, to be applied by the learner, as occasion requires, to the solutions of the miscellaneous problems given at the end of the present article.

PROBLEM I.

The base, and the sum of the hypothenuse and perpendicular of a right angled triangle being given, it is required to determine the triangle.



Let ABC, right angled at C, be the proposed triangle; and put BC$=b$ and AC$=x$.

Then, if the sum of AB and AC be represented by s, the hypothenuse AB will be expressed by $s-x$.

But, by the well known property of right angled triangles (Euc. I. 47) . . . . (l)

$$AC^2+BC^2=AB^2, \text{ or}$$
$$x^2+b^2=s^2-2sx+x^2.$$

Whence, omitting x^2, which is common to both sides of the equation, and transposing the other terms, we shall have

$$2sx=s^2-b^2, \text{ or}$$
$$x=\frac{s^2-b^2}{2s},$$

which is the value of the perpendicular AC; where s and b may be any numbers whatever, provided s be greater than b.

In like manner, if the base and the difference between the hypothenuse and perpendicular be given, we shall have, by putting x for the perpendicular and $d+x$ for the hypothenuse,

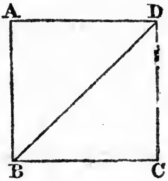
$$x^2+2dx+d^2=b^2+x^2, \text{ or}$$
$$x=\frac{b^2-d^2}{2d}.$$

(l) The edition of EUCLID, referred to in this and all the following problems, is that of Dr. SIMSON, London, 1801; which may also be used in the geometrical construction of these problems, should the student be inclined to exercise his talents upon this elegant, but more difficult branch of the subject.

Where the base (b) and the given difference (d) may be any numbers as before, provided b be greater than d.

PROBLEM II.

The difference between the diagonal of a square and one of its sides being given, to determine the square.



Let AC be the proposed square, and put the side BC, or CD, $=x$.

Then, if the difference of BD and BC be put $=d$, the hypothenuse BD will be $=x+d$.

But since, as in the former problem, BC^2+CD^2, or $2BC^2=BD^2$, we shall have

$$2x^2=x^2+2dx+d^2, \text{ or}$$
$$x^2-2dx=d^2.$$

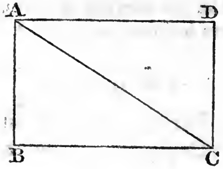
Which equation, being resolved according to the rule laid down for quadratics, in the preceding part of the work, gives

$$x=d+d\sqrt{2}.$$

Which is the value of the side BC, as was required.

PROBLEM III.

The diagonal of a rectangle ABCD, and the perimeter, or sum of all its four sides, being given, to find the sides.



Let the diagonal AC$=d$, half the perimeter AB$+$BC $=a$, and the base BC$=x$; then will the altitude AB$=$ $a-x$.

And since, as in the former problem, AB$^2+$BC$^2=$AC^2, we shall have

$$a^2-2ax+x^2+x^2=d^2, \text{ or}$$
$$x^2-ax=\frac{d^2-a^2}{2}.$$

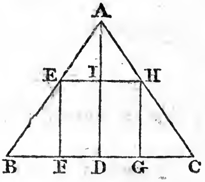
Which last equation, being resolved, gives

$$x=\tfrac{1}{2}a\pm\tfrac{1}{2}\sqrt{(2d^2-a^2)}.$$

Where a must be taken greater than d and less than $d\sqrt{2}$.

PROBLEM IV.

The base and perpendicular of any plane triangle ABC being given, to find the side of its inscribed square.



Let EG be the inscribed square; and put BC$=b$, AD$=p$, and the side of the square EH or EF$=x$.

Then, because the triangles ABC, AEH, are similar, (Euc. VI. 4,) we shall have

$$AD : BC :: AI : EH, \text{ or}$$
$$p : b :: (p-x) : x.$$

Whence, taking the products of the means and extremes, there will arise

$$px = bp - bx.$$

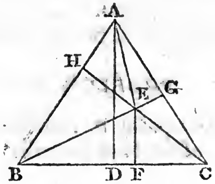
Which, by transposition and division, gives

$$x = \frac{bp}{b+p}.$$

Where b and p may be any numbers whatever, either whole or fractional.

PROBLEM V.

Having the lengths of three perpendiculars, EF, EG, EH, drawn from a certain point E, within an equilateral triangle ABC, to its three sides, to determine the sides



Draw the perpendicular AD, and having joined EA, EB, and EC, put $EF = a$, $EG = b$, $EH = c$, and BD (which is $\frac{1}{2}BC) = x$.

Then, since AB, BC, or CA, are each $= 2x$, we shall have, by Euc. I, 47,

$$AD = \sqrt{(AB^2 - BD^2)} = \sqrt{(4x^2 - x^2)} = \sqrt{3x^2} = x\sqrt{3}.$$

And because the area of any plane triangle is equal to half the rectangle of its base and perpendicular, it follows, that,

$$\triangle\, ABC = \tfrac{1}{2}BC \times AD = x \times x\sqrt{3} = x^2\sqrt{3},$$
$$\triangle\, BEC = \tfrac{1}{2}BC \times EF = x \times a \quad = ax,$$
$$\triangle\, AEC = \tfrac{1}{2}AC \times EG = x \times b \quad = bx,$$
$$\triangle\, AEB = \tfrac{1}{2}AB \times EH = x \times c \quad = cx,$$

But the last three triangles, BEC, AEC, AEB, are together, equal to the whole triangle ABC; whence

$$x^2\sqrt{3} = ax + bx + cx.$$

And, consequently, if each side of this equation be divided by x, we shall have

$$x\sqrt{3} = a + b + c, \text{ or}$$

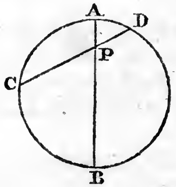
$$x = \frac{a+b+e}{\sqrt{3}}.$$

Which is, therefore, half the length of either of the three equal sides of the triangle.

Cor. Since, from what is above shewn, AD is $= x\sqrt{3}$, it follows, that the sum of all the perpendiculars, drawn from any point in an equilateral triangle to each of its sides, is equal to the whole perpendicular of the triangle.

PROBLEM VI.

Through a given point P, in a given circle ACBD, to draw a cord CD, of a given length.



Draw the diameter APB; and put CD $= a$, AP $= b$, PB $= c$, and CP $= x$; then will PD $= a - x$.

But, by the property of the circle, (Euc. III. 35,) CP × PD = AP × PB; whence

$$x(a-x)=bc, \text{ or}$$
$$x^2-ax=-bc.$$

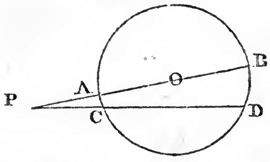
Which equation, being resolved in the usual way, gives

$$x=\tfrac{1}{2}a\pm\sqrt{(\tfrac{1}{4}a^2-bc)};$$

Where x has two values, both of which are positive.

PROBLEM VII.

Through a given point P, without a given circle ABDC, to draw a right line so that the part CD, intercepted by the circumference, shall be of a given length.



Draw PAB through the center O; and put CD $= a$ PA $=b$, PB $=c$, and PC $=x$; then will PD $=x+a$.

But, by the property of the circle, (Euc. III, 36, cor.,) PC × PD = PA × PB; whence

$$x(x+a)=bc, \text{ or}$$
$$x^2+ax=bc.$$

Which equation being resolved, as in the former problem, gives

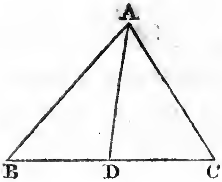
$$x=-\tfrac{1}{2}a\pm\sqrt{(\tfrac{1}{4}a^2+bc)};$$

Where one value of x is positive and the other negative. (m)

(m) The two last problems, with a few slight alterations, may be readily employed for finding the roots of quadratic equations by construction; but this, as well as the methods

PROBLEM VIII.

The base BC, of any plane triangle ABC, the sum of the sides AB, AC, and the line AD, drawn from the vertex to the middle of the base, being given, to determine the triangle.



Put BD or DC$=a$, AD$=b$, AB$+$AC$=s$, and AB$=x$; then will AC$=s-x$.

But, by my Geometry, B. II, 19, $\text{AB}^2+\text{AC}^2=\text{BD}^2+2\text{AD}^2$; whence

$$x^2+(s-x)^2=2a^2+2b^2, \text{ or}$$
$$x^2-sx=a^2+b^2-\tfrac{1}{2}s^2.$$

Which last equation, being resolved as in the former instances, gives

$$x=\tfrac{1}{2}s\pm\sqrt{(a^2+b^2-\tfrac{1}{4}s^2)},$$

for the values of the two sides AB and AC of the triangle; taking the sign $+$ for one of them, and $-$ for the other, and observing that a^2+b^2 must be greater than $\frac{1}{4}s^2$.

PROBLEM IX.

The two sides AB, AC, and the line AD, bisecting the vertical angle of any plane triangle, ABC, being given, to find the base BC.

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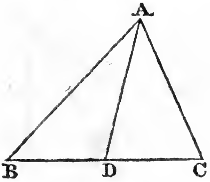
frequently given for constructing cubic and some of the higher orders of equations, is a matter of little importance in the present state of mathematical science; analysis, in these cases, being generally thought a more commodious instrument than geometry.



Put AB$=a$, AC$=b$, AD$=c$, and BC$=x$; then, by Euc. VI. 3, we shall have

$$AB(a) : AC(b) :: BD : DC.$$

And, consequently, by the composition of ratios (Euc. V, 18,)

$$a+b : a :: x : BD=\frac{ax}{a+b},$$

and

$$a+b : b :: x : DC=\frac{bx}{a+b}.$$

But, by Euc. VI, 13, BD$\times$DC$+$AD$^2=$AB$\times$AC; wherefore, also,

$$\frac{abx^2}{(a+b)^2}+c^2=ab, \text{ or}$$

$$abx^2=(a+b)^2\times(ab-c^2).$$

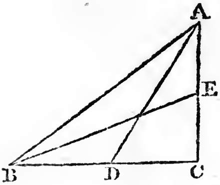
From which last equation we have

$$x=(a+b)\sqrt{\frac{ab-c^2}{ab}};$$

Which is the value of the base BC, as required.

PROBLEM X.

Having given the lengths of two lines, AD, BE, drawn from the acute angles of a right-angled triangle ABC, to the middle of the opposite sides, it is required to determine the triangle.



Put AD$=a$, BE$=b$, CD or $\frac{1}{2}$CB$=x$, and CE or $\frac{1}{2}$CA$=y$; then, since (Euc. I, 47) CD$^2+$CA$^2=$AD^2, and CE$^2+$CB$^2=$BE^2, we shall have

$$x^2+4y^2=a^2,$$
$$y^2+4x^2=b^2,$$

Whence, taking the second of these equations from four times the first, there will arise

$$15y^2=4a^2-b^2, \text{ or}$$
$$y=\sqrt{\frac{4a^2-b^2}{15}}.$$

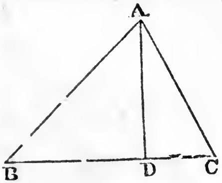
And, in like manner, taking the first of the same equations from four times the second, there will arise

$$15x^2=4b^2-a^2, \text{ or}$$
$$x=\sqrt{\frac{4b^2-a^2}{15}}.$$

Which values of x and y are half the lengths of the base and perpendicular of the triangle; observing that b must be less than $2a$, and greater than $\frac{1}{2}a$.

PROBLEM XI.

Having given the ratio of the two sides of a plane triangle ABC, and the segments of the base, made by a perpendicular falling from the vertical angle, to determine the triangle.



Put BD$=a$, DC$=b$, AB$=x$, AC$=y$, and the ratio of the sides as m to n.

Then, since, by the question, AB : AC :: m : n, and by B. II, 16, of my *Elements of Geometry*, AB$^2-$AC$^2=$ BD$^2-$DC^2, we shall have

$$x : y :: m : n, \text{ and}$$
$$x^2-y^2=a^2-b^2.$$

But, by the first of these expressions, $nx=my$, or $y=\frac{nx}{m}$; whence, if this be substituted for y in the second, there will arise

$$x^2-\frac{n^2}{m^2}x^2=a^2-b^2, \text{ or}$$
$$(m^2-n^2)x^2=m^2(a^2-b^2)$$

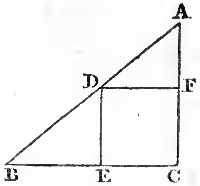
And, consequently, by division and extracting the square root, we shall have

$$x=m\sqrt{\frac{a^2-b^2}{m^2-n^2}}, \text{ and}$$
$$y=n\sqrt{\frac{a^2-b^2}{m^2-n^2}};$$

which are the values of the two sides AB, AC, of the triangle, as was required.

PROBLEM XII.

Given the hypothenuse of a right-angled triangle ABC, and the side of its inscribed square DC, to find the other two sides of the triangle.



Put AB $=h$, DE, or DF $=s$, AC $=x$, and CB $=y$; then, by similar triangles, we shall have

$$\text{AC}(x) : \text{CB}(y) :: \text{AF}(x-s) : \text{FD}(s).$$

And, consequently, by multiplying the means and extremes,

$$xy-sy=sx, \text{ or}$$
$$xy=s(x+y), \quad . \; . \; . \; (1)$$

But since, by Euc. I, 47, $\text{AC}^2+\text{CB}^2=\text{AB}^2$, we shall likewise have

$$x^2+y^2=h^2. \quad . \; . \; . \; . \; . \; . \; (2)$$

Whence, if twice equation (1) be added to equation (2), there will arise

$$x^2+2xy+y^2=h^2+2s(x+y), \text{ or}$$
$$(x+y)^2-2s(x+y)=h^2.$$

Which equation, being resolved after the manner of a quadratic, gives

$$x+y=s\pm\sqrt{(h^2+s^2)}, \text{ or}$$
$$y=s-x\pm\sqrt{(h^2+s^2)}.$$

Hence, if this value be substituted for y in equation (1), there will arise

$$x\{s-x\pm\sqrt{(h^2+s^2)}\}=s\{s\pm\sqrt{(h^2+s^2)}\}, \text{ or}$$
$$x^2-\{s\pm\sqrt{(h^2+s^2)}\}x=-s\{s\pm\sqrt{(h^2+s^2)}\}.$$

And, consequently, by resolving this last equation, we shall have

$$x=\tfrac{1}{2}\{s\pm\sqrt{(h^2+s^2)}\}\pm\sqrt{\{\tfrac{1}{4}h^2-\tfrac{1}{2}s^2\mp\frac{s}{2}\sqrt{(h^2+s^2)}\}}$$

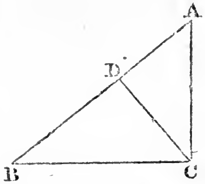
and

$$y=\tfrac{1}{2}\left\{s\pm\sqrt{(h^2+s^2)}\right\}\mp\sqrt{\left\{\tfrac{1}{4}h^2-\tfrac{1}{2}s^2\mp\frac{s}{2}\sqrt{(h^2+s^2)}\right\}}$$

Which are the values of the perpendicular AC and base BC, as was required.

PROBLEM XIII.

Having given the perimeter of a right-angled triangle ABC, and the perpendicular CD, falling from the right angle on the hypothenuse, to determine the triangle.



Put p=perimeter, CD$=a$, AC$=x$, and BC$=y$; then AB$=p-(x+y)$.

But, by right-angled triangles (Euc. I, 47) $AC^2+BC^2=AB^2$; whence

$$x^2+y^2=p^2-2p(x+y)+x^2+2xy+y^2.$$

Or, by transposing the terms and dividing by 2,

$$p(x+y)-\tfrac{1}{2}p^2=xy. \quad . \; . \; . \; . \; . \quad (1)$$

And since, by similar triangles, AB : BC :: AC : CD, we shall also have, by multiplying the means and extremes, $AB\times CD=BC\times AC$, or

$$ap-(ax+y)=xy. \quad . \; . \; . \; . \; . \quad (2)$$

Whence, by comparing equation (1) with equation (2), there will arise

$$(a+p)\times(x+y)=ap+\tfrac{1}{2}p^2.$$

Where

$$x+y=\frac{p(a+\frac{1}{2}p)}{a+p}, \text{ or}$$

$$y=\frac{p(a+\frac{1}{2}p)}{a+p}-x.$$

And if these values be now substituted for $x+y$ and y in equation (2), the result, when simplified and reduced, will give

$$(a+p)x^2-p(a+\tfrac{1}{2}p)x=-\tfrac{1}{2}ap^2.$$

From which last equation and the value of y, above found, we shall have

$$x \text{ or AC}=\frac{p(a+\frac{1}{2}p)}{2(a+p)}\pm\frac{p}{2(a+p)}\sqrt{\{(a-\tfrac{1}{2}p)^2-2a^2\}}$$

and

$$y \text{ or BC}=\frac{p(a+\frac{1}{2}p)}{2(a+p)}\mp\frac{p}{2(a+p)}\sqrt{\{(a-\tfrac{1}{2}p)^2-2a^2\}}$$

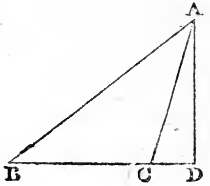
And, if the sum of these two sides be taken from p, the result will give

$$\text{AB}=p-(x+y)=\frac{p^2}{2(a+p)}.$$

Which expressions are, therefore, respectively equal to the values of the three sides of the triangle.

PROBLEM XIV.

Given the perpendicular, base, and sum of the sides of an obtuse-angled plane triangle ABC, to determine the two sides of the triangle.



Let the perpendicular AD$=p$, the base BC$=b$, the sum of AB and AC$=s$, and their difference $=x$.

Then, since half the difference of any two quantities added to half their sum gives the greater, and, when, subtracted, the less, we shall have

$$AB=\tfrac{1}{2}(s+x),\text{ and } AC=\tfrac{1}{2}(s-x).$$

But, by Euc. I, 47, $CD^2=AC^2-AD^2$, or $CD=\sqrt{\{\tfrac{1}{4}(s-x)^2-p^2\}}$; and, by B. II, 12, $AB^2=BC^2+AC^2+2BC\times CD$; whence

$$\tfrac{1}{4}(s+x)^2=b^2+\tfrac{1}{4}(s-x)^2+2b\sqrt{\{\tfrac{1}{4}(s-x)^2-p^2\}},\text{ or}$$
$$sx-b^2=2b\sqrt{\{\tfrac{1}{4}(s-x)^2-p^2\}}.$$

And if each of the sides of this last equation be squared, there will arise by transposition and simplifying the result,

$$(s^2-b^2)x^2=b^2(s^2-b^2)-4b^2p^2,\text{ or}$$
$$x=b\sqrt{(1-\frac{4p^2}{s^2-b^2})}.$$

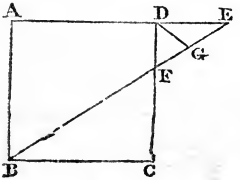
Whence, by addition and subtraction, we shall have

$$AB=\frac{s}{2}+\frac{b}{2}\sqrt{(1-\frac{4p^2}{s^2-b^2})},\text{ and}$$
$$AC=\frac{s}{2}-\frac{b}{2}\sqrt{(1-\frac{4p^2}{s^2-b^2})}.$$

Which are the sides of the triangle, as was required.

PROBLEM XV.

It is required to draw a right line BFE from one of the angles B of a given square BD, so that the part FE, intercepted by DE and DC, shall be of a given length.



Bisect FE in G, and put AB or BC$=a$, FG or GE$=b$, and BG$=x$; then will BE$=x+b$ and BF$=x-b$.

But since, by right-angled triangles, $AE^2=BE^2-AB^2$, we shall have

$$AE=\sqrt{\{(x+b)^2-a^2\}}.$$

And, because the triangles, BCF, EAB, are similar,

$$BF : BC :: BE : AE, \text{ or}$$

$$a(x+b)=(x-b)\sqrt{\{(x+b)^2-a^2\}}$$

Whence, by squaring each side of this equation, and arranging the terms in order, there will arise

$$x^4-2(a^2+b^2)x^2=b^2(2a^2-b^2).$$

Which equation, being resolved after the manner of a quadratic, will give

$$x=\sqrt{\{a^2+b^2\pm a\sqrt{(a^2+4b^2)}\}}$$

And, consequently, by adding b to, or subtracting it from this last expression, we shall have

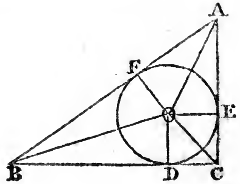
$$BE=\sqrt{\{a^2+b^2\pm a\sqrt{(a^2+4b^2)}\}}+b, \text{ or}$$
$$BF=\sqrt{\{a^2+b^2\pm a\sqrt{(a^2+4b^2)}\}}-b.$$

Which values, by determining the point E, or F, will satisfy the problem.

Where it may be observed, that the point G lies in the circumference of a circle, described from the centre D, with the radius FG, or half the given line.

PROBLEM XVI.

The perimeter of a right-angled triangle ABC, and the radius of its inscribed circle being given, to determine the triangle.



Let the perimeter of the triangle $=p$, the radius OD OE, or of the inscribed circle $=r$, AE$=x$, and BD$=y$.

Then, since in the right-angled triangles AEO, AFO, OE is equal to OF, and AO is common, AF will also be equal AE, or x.

And, in like manner, it may be shown, that BF is equal to BD, or y.

But, by the question, and Euc. I, 47, we have

$$(x+r)+(y+r)+(x+y)=p, \text{ and}$$
$$(x+r)^2+(y+r)^2=(x+y)^2.$$

Or, by adding the terms of the first, and squaring those of the second,

$$x+y=\tfrac{1}{2}p-r, \text{ and}$$
$$r(x+y)=xy-r^2.$$

Hence, since, in the first of these equations, $y=(\frac{1}{2}p-r)-x$, if this value be substituted for y in the second, there will arise

$$x^2-(\tfrac{1}{2}p-r)x=-\tfrac{1}{2}pr.$$

Which equation, being resolved in the usual manner, gives

$$x=\tfrac{1}{2}(\tfrac{1}{2}p-r)\pm\surd\{\tfrac{1}{4}(\tfrac{1}{2}p-r)^2-\tfrac{1}{2}pr.\},$$
and
$$y=\tfrac{1}{2}(\tfrac{1}{2}p-r)\mp\surd\{\tfrac{1}{4}(\tfrac{1}{2}p-r)^2-\tfrac{1}{2}pr.\},$$

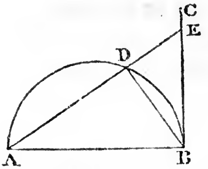
And, consequently, if r be added to each of these last expressions, we shall have

$$\text{AC}=\tfrac{1}{2}(\tfrac{1}{2}p+r)\pm\surd\{\tfrac{1}{4}(\tfrac{1}{2}p-r)^2-\tfrac{1}{2}pr.\},$$
and
$$\text{BC}=\tfrac{1}{2}(\tfrac{1}{2}p+r)\mp\surd\{\tfrac{1}{4}(\tfrac{1}{2}p-r)^2-\tfrac{1}{2}pr.\},$$

for the values of the perpendicular and base of the triangle, as was required.

PROBLEM XVII.

From one of the extremities A, of the diameter of a given semicircle ADB, to draw a right line AE, so that the part DE, intercepted by the circumference and a perpendicular drawn from the other extremity, shall be of a given length.



Let the diameter $AB = d$, $DE = a$, and $AE = x$; and join BD.

Then, because the angle ADB is a right angle, (Euc. III, 31,) the triangles ABE, ABD, are similar.

And, consequently, by comparing their like sides, we shall have

$$AE : AB :: AB : AD, \text{ or}$$
$$x : d :: d : x - a.$$

Whence, multiplying the means and extremes of these proportionals, there will arise

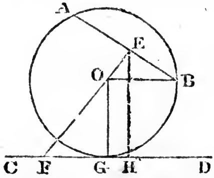
$$x^2 - ax = d^2.$$

Which equation, being resolved after the usual manner, gives

$$x = \tfrac{1}{2}a + \sqrt{(\tfrac{1}{4}a^2 + d^2)}.$$

PROBLEM XVIII.

To describe a circle through two given points, A, B, that shall touch a right line CD given in position.



Join AB; and through O, the assumed centre of the required circle, draw FE perpendicular to AB; which will bisect it in E (Euc. III, 3).

Also, join OB; and draw EH, OG, perpendicular to CD;

the latter of which will fall on the point of contact G (Euc. III, 18).

Hence, since A, E, B, H, F, are given points, put $EB=a$, $EF=b$, $EH=c$, and $EO=x$; which will give $OF=b-x$.

Then, because the triangle OEB is right-angled at E, we shall have

$$OB^2=EO^2+EB^2, \text{ or}$$
$$OB=\sqrt{(x^2+a^2)}.$$

But, by similar triangles, FE : EH :: FO : OG or OB or $b : c :: b-x :$ OB; whence, also,

$$OB=\frac{c}{b}(b-x).$$

And, consequently, if these two values of OB be put equal to each other, there will arise

$$\sqrt{(x^2+a^2)}=\frac{c}{b}(b-x).$$

Or, by squaring each side of this equation, and simplifying the result,

$$(b^2-c^2)x^2+2bc^2x=b^2(c^2-a^2.)$$

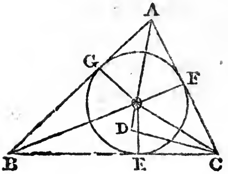
Which last equation, when resolved in the usual manner, gives

$$x=-\frac{bc^2}{b^2-c^2}+b\sqrt{\left\{\frac{c^4}{(b^2-c^2)^2}+\frac{c^2-a^2}{b^2-c^2}\right\}},$$

for the distance of the centre O from the chord AB; where b must, evidently, be greater than c, and c greater than a.

PROBLEM XIX.

The three lines AO, BO, CO, drawn from the angular points of a plane triangle ABC, to the centre of its inscribed circle, being given, to find the radius of the circle, and the sides of the triangle.



Let O be the centre of the circle, and, on AO produced, let fall the perpendicular CD; and draw OE, OF, OG, to the points of contact E, F, G.

Then, because the three angles of the triangle ABC are, together, equal to two right angles, (Euc. I, 32,) the sum of their halves OAC + OCA + OBE will be equal to one right angle.

But the sum of the two former of these, OAC + OCA, is equal to the external angle DOC; whence the sum of DOC + OBE, as also of DOC + OCD, is equal to a right angle; and, consequently, OBE = OCD.

Let, therefore, AO $= a$, BO $= b$, CO $= c$, and the radius OE, OF or OG $= x$.

Then, since the triangles BOE, COD are similar, BO : OE :: CO : OD, or $b : x :: c :$ OD; which gives

$$\text{OD} = \frac{cx}{b}, \text{ and CD} = \surd\left(c^2 - \frac{c^2x^2}{b^2}\right) \text{ or } \frac{c}{b}\surd(b^2 - x^2).$$

Also, because the triangle AOC is obtuse-angled at O, we shall have (Euc. II, 12)

$$\text{AC}^2 = \text{AO}^2 + \text{CO}^2 + 2\text{AO} \times \text{OD}; \text{ or}$$

$$\text{AC} = \surd\left(a^2 + c^2 + \frac{2acx}{b}\right) \text{ or } \surd\left(\frac{b(a^2+c^2)+2acx}{b}\right)$$

But the triangles ACD, AOF, being likewise similar,

AC : CD :: AO : OF, or

$$\surd\left(\frac{b(a^2+c^2)+2acx}{b}\right) : \frac{c}{b}\surd(b^2 - x^2) :: a : x.$$

Whence, multiplying the means and extremes, and squaring the result, there will arise

$$bx^2\{b(a^2+c^2)+2acx\}=a^2c^2(b^2-x^2).$$

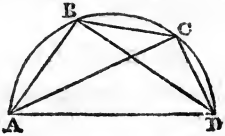
Or, by collecting the terms together, and dividing by the coefficient of the highest power of x,

$$x^3+\left(\frac{ab}{2c}+\frac{ac}{2b}+\frac{bc}{2a}\right)x^2=\frac{abc}{2}.$$

From which last equation x may be determined, and thence the sides of the triangle. (d)

PROBLEM XX.

Given the three sides AB, BC, CD, of a trapezium ABCD, inscribed in a semicircle, to find the diameter, or remaining side AD.



Let AB $=a$, BC $=b$, CD $=c$, and AD $=x$; then, by Euc. VI, D, AC $\times$ BD $=$ AD $\times$ BC $+$ AB $\times$ CD $=bx+ac$.

But ABD, ACD, being right angles, (Euc. III, 31,) we shall have

$$\text{AC}=\sqrt{(\text{AD}^2-\text{DC}^2)}, \text{ or } \sqrt{(x^2-c^2)}, \text{ and}$$
$$\text{BD}=\sqrt{(\text{AD}^2-\text{AB}^2)}, \text{ or } \sqrt{(x^2-a^2)}.$$

Whence, by substituting these two values in the former expression, there will arise

---

(d) This, and the following problem, cannot be constructed geometrically, or by means only of right lines and a circle, being what the ancients usually denominated solid problems, from the circumstance of their involving an equation of more than two dimensions; in which cases they generally employed the conic sections, or some of the higher orders of curves.

$$\sqrt{(x^2-c^2)} \times \sqrt{(x^2-a^2)} = bx + ac.$$

Or, by squaring each side, and reducing the result

$$x^3 - (a^2+b^2+c^2)x = 2abc.$$

From which last equation the value of x may be found, as in the last problem. (n)

MISCELLANEOUS PROBLEMS.

PROBLEM I.

To find the side of a square, inscribed in a given semicircle, whose diameter is d. Ans. $\frac{1}{5}d\sqrt{5}$

PROBLEM II.

Having given the hypothenuse (13) of a right-angled triangle, and the difference between the other two sides (7), to find these sides. (o) Ans. 5 and 12

PROBLEM III.

To find the side of an equilateral triangle, inscribed in a circle, whose diameter is d; and that of another circumscribed about the same circle.

Ans. $\frac{1}{2}d\sqrt{3}$, and $d\sqrt{3}$

PROBLEM IV.

To find the side of a regular pentagon, inscribed in a circle, whose diameter is d. Ans. $\frac{1}{4}d\sqrt{(10-2\sqrt{5})}$

(n) Newton, in his *Universal Arithmetic*, English edition, 1728, has resolved this problem in a variety of different ways, in order to show, that some methods of proceeding, in cases of this kind, frequently lead to more elegant solutions than others; and that a ready knowledge of these can only be obtained by practice.

(o) Such of these questions as are proposed in numbers, should first be resolved generally, by means of the usual symbols, and then reduced to the answers above given, by substituting the numeral values of the letters in the results thus obtained.

PROBLEM V.

To find the sides of a rectangle, the perimeter of which shall be equal to that of a square, whose side is a, and its area half that of the square.

Ans. $a+\frac{1}{2}a\sqrt{2}$ and $a-\frac{1}{2}a\sqrt{2}$

PROBLEM VI.

Having given the side (10) of an equilateral triangle, to find the radii of its inscribed and circumscribing circles. Ans. 2.8868 and 5.7736

PROBLEM VII.

Having given the perimeter (12) of a rhombus, and the sum (8) of its two diagonals, to find the diagonals.

Ans. $4+\sqrt{2}$ and $4-\sqrt{2}$

PROBLEM VIII.

Required the area of a right-angled triangle, whose hypothenuse is x^{3x}, and the base and perpendicular x^{2x} and x^{x}. Ans. 1.029085

PROBLEM IX.

Having given the two contiguous sides (a, b) of a parallelogram, and one of its diagonals (d), to find the other diagonal. Ans. $\sqrt{(2a^2+2b^2-d^2)}$

PROBLEM X.

Having given the perpendicular (300) of a plane triangle, the sum of the two sides (1150), and the difference of the segments of the base (495), to find the base and the sides. Ans. 945, 375, and 780

PROBLEM XI.

The lengths of three lines drawn from the three angles of a plane triangle to the middle of the opposite sides, being 18, 24, and 30, respectively; it is required to find the sides. Ans. 20, 28.844, and 34.176

PROBLEM XII.

In a plane triangle, there is given the base (50), the area (796), and the difference of the sides (10), to find the sides and the perpendicular.

Ans. 36, 46, and 33.261

PROBLEM XIII.

Given the base (194) of a plane triangle, the line that bisects the vertical angle (66), and the diameter (200) of the circumscribing circle, to find the other two sides. Ans. 81.36587 and 157.43865

PROBLEM XIV.

The lengths of two lines that bisect the acute angles of a right-angled plane triangle, being 40 and 50 respectively, it is required to determine the three sides of the triangle. Ans. 35.80737, 47.40728, and 59.41143

PROBLEM XV.

Given the altitude (4), the base (8), and the sum of the sides (12), of a plane triangle, to find the sides.

Ans. $6+\frac{4}{5}\sqrt{5}$ and $6-\frac{4}{5}\sqrt{5}$

PROBLEM XVI.

Having given the base of a plane triangle (15,) its area (45), and the ratio of its other two sides as 2 to 3, it is required to determine the lengths of these sides.

Ans. 7.7915 and 11.6872

PROBLEM XVII.

Given the perpendicular (24), the line bisecting the base (40), and the line bisecting the vertical angle (25) to determine the triangle.

Ans. The base $\frac{250}{7}\sqrt{7}$

From which the other two sides may be readily found.

PROBLEM XVIII.

Given the hypothenuse (10) of a right-angled triangle, and the difference of two lines drawn from its extremities to the centre of the inscribed circle (2), to determine the base and perpendicular. Ans. 8.08004 and 5.87447

PROBLEM XIX.

Having given the lengths (a, b,) of two chords, cutting each other at right angles, in a circle, and the distance (c) of their point of intersection from the centre, to determine the diameter of the circle.

Ans. $\sqrt{\{\frac{1}{2}(a^2+b^2)+2c^2\}}$

PROBLEM XX.

Two trees, standing on an horizontal plane, are 120 feet asunder; the height of the highest of which is 100 feet, and that of the shortest 80; whereabout in the plane must a person place himself, so that his distance from the top of either of the trees, shall be equal to the distance between them?

Ans. $20\sqrt{21}$ feet from the bottom of the shortest, and $40\sqrt{3}$ feet from the bottom of the other.

PROBLEM XXI.

Having given the sides of a trapezium, inscribed in a circle, equal to 6, 4, 5, and 3, respectively, to determine the diameter of the circle.

Ans. $\frac{1}{20}\sqrt{(130\times 153)}$ or 7.051595

PROBLEM XXII.

Supposing the town A to be 30 miles from B, B 25 miles from C, and C 20 miles from A; whereabouts must a house be erected that it shall be at equal distance from each of them?

Ans. 15.118556 miles from each.

PROBLEM XXIII.

Given the area (100) of an equilateral triangle ABC, whose base BC falls on the diameter, and vertex A in the middle of the arc of a semicircle; required the diameter of the semicircle. Ans. $20\sqrt[4]{3}$

PROBLEM XXIV.

In a plane triangle, having given the perpendicular (p), and the radii (r, R,) of its inscribed and circumscribed circles, to determine the triangle.

Ans. The base $\dfrac{2r\sqrt{(2p\text{R}-4r\text{R}-r^2)}}{p-2r}$

PROBLEM XXV.

Having given the base of a plane triangle equal to $2a$, the perpendicular equal to a, and the sum of the cubes of its other two sides equal to three times the cube of the base; to determine the sides.

Ans. $a(2+\frac{1}{3}\sqrt{6})$ and $a(2-\frac{1}{3}\sqrt{6}$

ADDENDA.

A New Method of resolving Numerical Equations.

As the solution of equations by approximation is one of the most useful, and at the same time, one of the most tedious operations in modern algebra; several analysts of the first celebrity have turned their attention to this branch of mathematics. LAGRANGE has written a complete work on the subject; and if the method which he has there proposed, were as practicable as it is beautiful and complete in theory, nothing further could possibly be desired. But unfortunately the number of operations to be performed is so great, that the certainty of the result by no means compensates for the labour of obtaining it, and recourse would always be had to some more expeditious, although less perfect instrument.

Of late, however, two methods have been proposed, nearly at the same time, by MESSRS. HOBROID and HORNER; which are possessed both of great facility and practical convenience, and are held, on that account, in deserved estimation. Of these, the latter appears

decidedly the most perfect, and is, without doubt, by far the best method of approximation that has hitherto been published; it is, nevertheless, open to two material objections, the analysis from which it is derived is too high for the subject*, and sufficient provision is not made for determining all the roots in succession.

For these reasons, I have been induced to propose the mode of solving numerical equations that is here treated of; and which possesses the advantage of finding all the roots, whether real or imaginary, by a continuous process; whilst its principles are the same as those commonly employed in the doctrines of algebraic equations.

From the theory of these, laid down in the body of the work, it appears that they are produced by the multiplication of certain simple factors, which when known, immediately give us all the roots of the equation: it is our present object to discover these factors by a process of division differing but little from that commonly used, and which may be illustrated as follows:

Let the expression

$$\underset{1}{A}x^3 + \underset{2}{A}x^2 + \underset{3}{A}x + \underset{4}{A}$$

be divided by $x - r$.

---

* In deriving his rules, MR. HORNER has unfortunately made use of an analysis much more transcendental than was required; they may be demonstrated from the simplest principles of the differential calculus; but it certainly is to be wished that the rules for approximating to algebraic equations, should be demonstrated from antecedent principles; more especially as it is only in the case of these equations that even MR. HORNER's method is of any use.

Proceeding by the usual method

$$x-r\{\underset{1}{A}x^3+\underset{2}{A}x^2+\underset{3}{A}x+\underset{4}{A}\}\underset{1}{A}x^2+(\underset{2}{A}+\underset{1}{A}r)x+(\underset{3}{A}+\underset{2}{A}r+\underset{1}{A}r^2)+\frac{\underset{4}{A}+\underset{3}{A}r+\underset{2}{A}r^2+\underset{1}{A}r^3}{x-r}$$

$$\underset{1}{A}x^3-\underset{1}{A}rx^2$$

$$(\underset{2}{A}+\underset{1}{A}r)x^2+\underset{3}{A}x$$

$$(\underset{2}{A}+\underset{1}{A}r)x^2-(\underset{2}{A}+\underset{1}{A}r)rx$$

$$(\underset{3}{A}+\underset{2}{A}r+\underset{1}{A}r^2)x+\underset{4}{A}$$

$$(\underset{3}{A}+\underset{2}{A}r+\underset{1}{A}r^2)r^2-(\underset{3}{A}+\underset{2}{A}r+\underset{1}{A}r^2)r$$

$$\underset{4}{A}+\underset{3}{A}r+\underset{2}{A}r^2+\underset{1}{A}r^3.$$

Where it appears that the several terms in the quotient, are performed by adding to the same power of x in the dividend, the coefficient of the preceding term in the quotient, multiplied by r. And this law will evidently hold good however many terms there may be in the dividend.

We might therefore have arranged the division as follows.

A_1	A_2	A_3	A_4
0	$P_1 r$	$P_2 r$	$P_3 r$
P_1	P_2	P_3	P_4

When P_1, P_2 *&c.* are the several coefficients of the power of x; the omission of this letter, greatly facilitating the process, as will be more distinctly seen in the following example:

Let the expression

$$x^3+4x^2+6x+10$$

be divided by $x-2$.

Arranging the coefficients, and proceeding as above,

1	4	6	10
0	2	12	36
1	6	18	46

And the result is

$$x^2+6x+18+\frac{46}{x-2}.$$

To adapt this mode of division to the object we have in view, it will be necessary to modify it, so as to proceed figure by figure, when r contains more than one: and this we may accomplish as follows:

Let r contain two figures, and be represented thus

$$r=r'+r''.$$

Then since from the preceding operations it appears that any coefficient P_n of the quotient, is equal to

$$A_n+P_{n-1}.r$$

it follows that

$$P_n=A_n+P_{n-1}.r=A_n+P_{n-1}.(r'+r'')$$

$$=A_n+P_{n-1}.r'+P_{n-1}.r''.$$

And as the two first terms of this result, or $\underset{n}{\mathrm{A}}+\underset{n-1}{\mathrm{P}}.r'$, are the same as before, with the exception of r' being substituted for r; it is evident that the division will commence in the same manner; that is to say, if

$$\underset{1}{\mathrm{A}}x^3+\underset{2}{\mathrm{A}}x^2+\underset{3}{\mathrm{A}}x+\underset{4}{\mathrm{A}}$$

is to be divided by $x-(r'+r'')$, the first part of the operation will stand thus

$$\begin{array}{cccc}
\underset{1}{\mathrm{A}} & \underset{2}{\mathrm{A}} & \underset{3}{\mathrm{A}} & \underset{4}{\mathrm{A}} \\
0 & \underset{1}{\mathrm{P}}r' & \underset{2}{\mathrm{P}}r' & \underset{3}{\mathrm{P}}r' \\
\hline
\underset{1}{\mathrm{P}} & \underset{2}{\mathrm{P}} & \underset{3}{\mathrm{P}} & \underset{4}{\mathrm{P}}.
\end{array}$$

The next step requires a little more attention: it is evident that $\underset{1}{\mathrm{P}}$, $\underset{2}{\mathrm{P}}$, *&c.* are less than the complete coefficients of the quotient, which we will term $\underset{1}{\mathrm{P}}'$ $\underset{2}{\mathrm{P}}'$, *&c.* by all that part of the latter, that depends on r''; and expressing $\underset{n}{\mathrm{P}}'$, as before, by

$$\underset{n}{\mathrm{P}}'=\underset{n}{\mathrm{A}}+\underset{n-1}{\mathrm{P}}'r'+\underset{n-1}{\mathrm{P}}'.r''$$

it appears that this part of $\underset{n}{\mathrm{P}}'$, may be separated into two others, one $\underset{n-1}{\mathrm{P}}'r''$, which has been wholly neglected; and the other consisting of that portion of $\underset{n-1}{\mathrm{P}}'r'$, which contains r'': now this last is evidently equal to $(\underset{n-1}{\mathrm{P}}'-\underset{n-1}{\mathrm{P}})r'$, for $\underset{n-1}{\mathrm{P}}'$ is the complete quotient, and $\underset{n-1}{\mathrm{P}}$ is the quotient when r', only, is taken into account, their difference, therefore must express that part of $\underset{n-1}{\mathrm{P}}'$, which depends on r''.

Whence, for the next step in the division, add up every column, as it is found, both without its first term, and with it; multiply the first of these sums by r', and the second by r'' and the results added to the next coefficient

in the preceding division, will give the new coefficient sought.

The operation is as follows:

A_1	A_2	A_3	A_4
0	P_1r'	P_2r'	P_3r'
P_1	P_2	P_3	P_4
0	Q_1r'	Q_2r'	Q_3r'
0	P'_1r''	P_2r''	P_3r''
Q_1	Q_2	Q_3	Q_4
P'_1	P'_2	P'_3	P'_4

Where Q_1, Q_2, &c. are the sums of the columns they stand under, without their first terms P_1, P_2, &c.

EXAMPLE.

Divide $x^3+3x^2+3x-140$, by $x-4.2$.

1	3		3		−140	
.	4		28		124	
1	7		31		−16	
.	0			8	8	96
.		2	1	44	6	648
0		2	2	24	15	608
1	7	2	33	24	−	392

And the result is

$$x^2+7.2x+33.24-\frac{.392}{x-4.2}$$

If the quantity r contains three figures, and is represented by

$$r = r' + r'' + r''' ;$$

we shall have in the same way

A_1	A_2	A_3	A_4
0	$P_1 r'$	$P_2 r'$	$P_3 r''$
P_1	P_2	P_3	P_4
0	$Q_1 r'$	$Q_2 r'$	$Q_3 r'$
0	$P'_1 r''$	$P_2 r''$	$P_3 r''$
Q_1	Q_2	Q_3	Q_4
P'_1	P'_2	P'_3	P'_4
0	$Q'_1 r'$	$Q'_2 r'$	$Q'_3 r'$
0	$Q'_2 r''$	$Q'_3 r''$	$Q'_3 r''$
0	$P''_1 r'''$	$P''_2 r'''$	$P''_3 r'''$
Q'_1	Q'_2	Q'_3	Q'_4
P''_1	P''_2	P''_3	P''_4

To apply what has been here said to the solution of equations, it is only necessary to observe, that if r is a root of the equation, the last column must converge to zero; for this last column is what remains after dividing by $x-r$, and r being a root, $x-r$ is a divisor of the given equation, and therefore, can leave no remainder.

From which we obtain this practical rule; having determined at what distance from the place of units, the first figure of the root stands, substitute in that place every integer successively, and divide as above,

until the greatest number is found that does not change the sign of the remainder in the last column; set this down as the first figure of the root: and proceed in a similar manner with every figure, until the root is determined with a sufficient accuracy; observing, that whenever nought enters into the result, it will be necessary to examine the preceding figure, in order to determine whether a smaller number would not have left a less remainder.

EXAMPLE I.

Required a root of the equation $x^3+60x^2+1000x-1000=0$

$r=.9455148243.$

1	60		1000		−1000	
		9	54	81	949	329
	60	9	1054	81	−50	671
		4		36	2	22624
	60	94		4376	42	291344
		5	2	4736	44	517584
	60	945	1057	2836	−6	153416
		5		45		2784825
	60	9455		2		123770
		1		304725	5	2879651
	60	94551		309425	5	5788246
		4824	1057	593025	−	5745914
				45		278505
				2		12376
				25		1545
				304727		5288119
				309452		5580545
1	60	945514824	1057	623970 2	−	0165369

Carried forward.

Brought forward.		1057	623970	2
1	60|945514824		9	
			609	
			618	
		1057	624588	
			243	
		1057	624831	
			48	
		1057	624879	
			1	
1	60|945514824	1057	624880	

– 0165369
5562
244
30
3
105762
111601
– 53768
2187
96
10
1
42304
44598
– 9170
432
16
2
8456
8906
– 264
9
210
219
– 45
42
– 3
3
.

Although the number of figures in this example is much greater than is necessary, as we shall see when we come to speak of the contractions that may be used, yet it affords a sufficient proof of the advantage

attending our method; the root is obtained by a process so simple as to be easily remembered, and the result not only gives the root in question, but also the coefficients of the reduced equation, which contains the other two roots; this equation is evidently.

$$x^2+60.945514824x+1057.62488=0.$$

In the next example we shall extract all the roots in one continuous operation, and this advantage will then be more distinctly seen.

Example II.

Required all the roots of the equation.

$$x^3-9x-9=0.$$

Here, it being evident that one root is nearly equal to 1, it will be advantageous to subtract unity from all the roots, which is accomplished by putting $x=z+1$ the resulting equation is

$$x^3+3x^2-6x+1=0$$

Whence by the rule.

1		3		−6		1
		1		31	−	569
	3	184791	−5	69		431
				8		2624
				2544	−	434208
			−	2624		23032
			−5	4276		13456
				4		10764
				32	−	216565
				12736	−	19234
			−	13456		3798
			−5	414144		

Carried forward.

Brought forward			−5	414144	3798
				7	235
				56	188
				2	9
				2226	−3787
				2354	−2355
			−5	411786	443
				9	30
				7	24
				286	−487
				302	433
			−5	411484	10
				3	
1	3	184791	−5	411483	
	1		4	184791	
	4	184791	−1	226692	
		2		2	
	4	384791		876958	
		2	1	076958	
	4	404791		149734	
		6		2	
	4	410791		4	
		6		88094	
	4	411391		112095	
		8		−37639	
	4	411471		6	
		4		12	
	4	411475		12	
				26464	
				33784	
				3855	

Carried forward.

$$
\begin{array}{rr}
\text{Brought forward} & 3855 \\
& 6 \\
& 12 \\
& 12 \\
& 3 \\
& \underline{2646} \\
& \underline{3381} \\
& -474 \\
& 8 \\
& 16 \\
& 1 \\
& \underline{352} \\
& \underline{449} \\
& -25 \\
& 4 \\
& \underline{17} \\
& \underline{21} \\
& -4
\end{array}
$$

And the three roots of the reduced equation are

$$
\begin{array}{r}
1.184791 \\
1.226684 \\
-4.411475
\end{array}
$$

from which those of the proposed equations may be obtained, by adding the unit that was subtracted.

These examples will sufficiently illustrate the rule for approximating to the real roots of equations; but when the imaginary roots are the objects of research, another and perhaps less commodious method must be employed. In this case the divisor is trinomial, and consequently, two figures are to be determined at each operation instead of one; which leaves a wider scope for ambiguity, and renders the tentative part of the process more fatiguing.

On this account the following method of extracting

quadratic divisors may require some further addition; but even in its present state, it is much more practicable and commodious than that given by LAGRANGE; which, as an instrument of calculation, may be considered as almost useless.

The theory is derived as before, from the common rules of division: for if

$$x^4 + A_1x^3 + A_2x^2 + A_3x + A_4$$

be divided by the quadratic divisor $x^2 - rx - s$, the operation will stand as follows.

$$x^2 - rx - 3\,\{x^4 + A_1x^3 + A_2x^2 + A_3x + A_4\}\; x^2 + (A_1 + r)\,x + \{(A_1 + s) + (A_2 + r)\,r\}$$

$$\frac{x^4 - rx^3 - sx^2}{(A_1 + r)\,x^3 + (A_2 + s)\,x^2 + A_3x}$$

$$\frac{(A_1 + r)\,x^3 - (A_1 + r)\,rx^2 - (A_1 + r)\,sx}{\{(A_1 + s) + (A_2 + r)r\}\,x^2 + \{A_3 + (A_1 + r)s\}\,x + A_4}$$

$$\frac{\{(A_1 + s) + (A_2 + r)r\}\,x^2 - \{(A_1 + s) + (A_2 + r)\,r\}\,rx - \{(A_1 + s) + (A_2 + r)\,r\}\,s}{\{A_3 + (A_1 + r)\,s + (A_1 + s)\,r + (A_2 + r)\,r^2\}\,x + \{A_4 + (A_1 + s)\,s + (A_2 + r)\,rs\}}$$

Where it is evident that any coefficient in the quotient, except the last, is obtained, by adding to that of the like power of x in the dividend, the penultimate coefficient in the quotient multiplied by r; and to that of the preceding power of x in the dividend, the antipenultimate coefficient in the quotient multiplied by s;

N 5

but in the last term, the preceding coefficient multiplied by r, is omitted.

Following this rule, the operation might be arranged thus

1	A_1	A_2	A_3	A_4
0	$P_1 r$	$P_2 r$	$P_3 r$	$P_3 s$
0	0	$P_1 s$	$P_2 s$	
P_1	P_2	P_3	P_4	P_5

When r and s contain more than one figure each, this rule may be modified, so as to proceed figure by figure, in a similar way to that before explained, and the formula will then be as follows.

A_1	A_2	A_3	A_4	A_5
.	$P_1 r'$	$P_1 s'$	$P_2 s'$	$P_3 s'$
.	.	$P_2 r'$	$P_3 r'$	
P_1	P_2	P_3	P_4	P_5
.	$P' r''$	$Q_1 r'$	$Q_1 s'$	$Q_2 s'$
.	.	$P_1 s''$	$Q_2 r'$	$P'_3 s''$
.	.	$P'_2 r''$	$P'_2 s''$	.
.	.	.	$P'_3 r''$	.
0	Q_1	Q_2	Q_3	Q_4
P'	P'_2	P'_3	P'_4	P'_5
0	$P'' r'''$	$Q'_1 r'$	$Q'_1 s'$	$Q'_2 s'$
	.	$Q'_1 r''$	$Q'_1 s''$	$Q'_2 s''$

Carried forward.

Brought forward.

.	.	$\underset{1}{P''}s'''$	$\underset{2}{Q'}r''$	$\underset{3}{P''}s'''$
.	.	$\underset{2}{P''}r'''$	$\underset{2}{Q'}r''$	.
.	.	.	$\underset{2}{P''}s'''$	.
.	.	.	$\underset{3}{P''}r'''$	.
o'	$\underset{1}{Q'}$	$\underset{2}{Q'}$	$\underset{3}{Q'}$	$\underset{4}{Q'}$
$\underset{1}{P''}$	$\underset{2}{P''}$	$\underset{3}{P''}$	$\underset{4}{P''}$	$\underset{5}{P''}$

The quantities $\underset{1}{Q}$, $\underset{2}{Q}$, &c., being, as before, the sums of the columns they stand under, omitting the first terms $\underset{2}{P}$, $\underset{3}{P}$, &c.; and the quantities $\underset{1}{P}$, $\underset{2}{P}$, $\underset{3}{P}$, &c. the sums of the same columns including the first terms.

Example.

Divide

$$x^4+3x^3+5\,x^2+7x+9$$

by the quadratic divisor

$$x^2-2.45x-3.46.$$

1	3		5		7		9	
.	2		3		15		54	
.	.		10		36		. .	
1	5		18		58		63	
		4		8	1	2	10	08
				4	6	72	8	544
			2	16	2	16		
					8	544		
1		4	3	36	18	624	18	624

Carried forward

Brought forward.

1	5	4	21	36	76	624	81	624
		05		10		15	1	3575
		05		2		2		1810
	5	45		6		905	1	30875
				27.25		181	2	84725
				45.25		327	84	47125
			21	81.25	1	090625		
					2	673625		
					79	297625		

When the remainders in the two last columns vanish, the trinomial is a complete divisor, and consequently, if the dividend is put under the form of an equation, the quadratic divisor will contain two of its roots; whence we have the following rule for finding a trinomial divisor to any given equation.

Determine at what distances from the place of units, the first figures in the coefficients of the binomial stand, and set down in those places, the greatest integers, that, when substituted for r' and s', in the preceding formula, leave the signs of the last columns unchanged. Proceed in a similar way with the next and succeeding figures, until the coefficients are determined with as much accuracy as is required. Observing that when nought enters into either of these numbers, the preceding figure should be examined, in order to try whether the next less digit, would not have left a smaller remainder.

EXAMPLE.

Find a trinomial divisor of the equation.

$$x^4-36x^2+72x-36=0.$$

$$r=2.1409$$
$$s=1.1067$$

1 0
2
2.1409

−36
−1
4
−33 |
 | 2
− | 1
 | 21
 | 31
−32 | 69
 | 8
 | 4
 | 856
 | 1696
−32 | 5204
 | −6
 | −6
−32 | 5264
 | 18
 | 9
 | 36
 | −7
 | 1926
 | 3152
−32 | 523248

72
−2
−66
4 |
− | 1
 | 62
− | 21
−3 | 269
−2 | 959
1 | 041
− | 04
 | 4
 | 3392
 | 1696
−1 | 300816
− | 988656
 | 052344
 | −12
 | −6
 | −24
 | −1284
 | −2568
 | 26664
 | −9
 | −9
 | −5
 | −1498
 | 6306
 | 315
 | 126
 | −29271
 | −25017
 | 1647

−36
33
−3 |
− | 31
3 | 269
2 | 959
− | 41
− | 1696
 | −1696
− | 22756
 | 6
 | 6
 | 195158
 | 201758
 | −25802
 | −3153
 | −315
 | −18
 | 22766
 | 19280
 | −6522

Whence the required trinomial is

$$x^2-2.1409x+1.1067.$$

This example sufficiently proves that no great difficulty is to be apprehended in guessing at the successive figures, as the imperfect divisors lead us to the correct integers in nearly every division; thus to approximate towards the value of r', we divide 72 by 36, the result of which, 2, is correct; to obtain the next figure, we divide 4 by 33, and the result 1 is also correct; and proceeding in the same way with the remaining figures, the required digits are always obtained within a unit of what they should be: to arrive at the first approximation to s', -36 is to be divided by 36, and the result -1, is the number sought; for the second figure, -3 is to be divided by 33, and -1.09, the quotient differs but little from $-.1$, the figure in the coefficient.

It cannot be expected that in every case, the successive figures will be so readily found, but after a little practice they may be guessed at without much difficulty; more especially if the equation be properly reduced before the operation is commenced.

It has been remarked in a former part of the addenda, that the process for extracting the real roots is capable of being abridged, and the same remark equally applies to the extraction of the imaginary roots; the nature of this abridgment, I shall now explain.

From the formula in page 267, it appears that

$$\underset{3}{P'}=\{\underset{2}{A}+\underset{1}{A}(2r'+r'')\}r''$$

$$\underset{3}{P''}=\{\underset{2}{A}+\underset{1}{A}(2r'+2r''+r''')\}r'''$$

$$\&c. \qquad \&c.$$

and, consequently, $\underset{3}{P'}$, $\underset{3}{P''}$, &c. may be found by doubling the addition which was made at the preceding step to

A_2, adding the last figure of the root, and multiplying by that figure: a process the same as that employed to extract the square root.

With this alteration, the example at page 270 would stand as follows.

1	3		−6	1
	.1		31	569
	3.184791	−5	69	431
			2624	2624
		−5	4276	45724
			13456	43428
		−5	414144	23032
			235	13456
		−5	41179	10764
			28	25454
1	3.184791	−5	41151	21656
	2	4	18479	3798
1	5.184791	−1	22672	235
	226689	1	07695	188
				9
	5.411480	−	14977	4230
			11209	3787
			-3768	443
			3378	28
			−390	22
			338	493
			52	486
			45	7
			7	

And the three roots are

1.184791
1.226689
−4.411480

It should be observed that in this example, after finding the root, the resulting quadratic is reduced to another whose roots are less by unity, with a view of facilitating the remaining process. Also, that in the third column, the additions being partly negative, and partly positive, the quantities with unlike signs have been added up separately; but it would be still better, in a case of this kind, to use the arithmetic complements as is done in logarithms.

It only remains for me to remark that the case in which the usual rules of approximation fail, is when the equation contains two roots that are nearly equal, in which instance the results approximate alternately to these roots; and if their number be even, no unit can be obtained by the usual criterion of the change of sign, since, the parts of the roots between these limits being equal, the results are always positive: and this difficulty will continue until the substitutions be pushed so far as to exhaust the figures which are common to the roots, an operation, in some cases, equivalent to that of resolving the equation.

In an instance of this kind, however, we may always be guided by the convergency of the results, unless the equation contains a pair of imaginary roots whose real parts, are nearly equal to the root sought. When this is the case, I am not aware of any method but that of LAGRANGE that can be employed, and it is only in this instance, and then too, when the equation is of small dimensions, that his beautiful, but impracticable theory, can be used with advantage.

ON THE SOLUTION OF EXPONENTIAL EQUATIONS.

IT has been observed in the body of the work, that no direct method exists for resolving the equation

$$x^x = a;$$

and the solution of numeral equations of this kind, is there obtained, by applying the rule of double position, to the results of successive substitutions.

But this very general process is liable to the same objections in this case, as in that of algebraic equations; the progress of the approximation is not seen, and the operations are not continuous.

Neither of these objections apply to the following very simple mode of solution; which however, unfortunately, does not extend beyond six figures, the range of the common tables of logarithms.

Since

$$x^x = a,$$

taking the logarithms on each side of the equation

$$x.lx = la:$$

and repeating the operation

$$lx + l^2x = l^2a.$$

or putting $lx = z$, and $l^2a = a'$

$$z + l.z = a'$$

Whence we have this rule.

Reduce the equation as above, and seek in the tables, the greatest number, which, added to its logarithm, (both being taken to one place only,) will produce a sum less than the first significant figure in a'. Subtract this sum from a' and call the result a''; seek, in like manner, the second figure of the number, which, added to the second figure of its logarithm, does not produce a greater sum than the first significant figure in a''. Subtract this sum from a''; and proceeding as before, we shall at length obtain a number, which when added to its logarithm, gives a result equal to a'; this number is the logarithm of x.

Example.

Required a root of the equation

$$x^x = 100.$$

Here by the given formula

$$z+l.z=.3010300$$

Let $$z=z'+z''+z'''+\&c.$$

$$l.z=l'+l''+l'''+\&c.$$

Then in the present case $z'=.5$, (for $.6+\log(.6)=.6+\bar{1}.77=.37$ is evidently too great,) and proceeding according to the rule.

			.3010300
$z^1+l^1=$	5 +	$\bar{1}.6=$	.1
			2010300
$z^2+l^2=$	5 +	14=	19
			110300
$z^3+l^3=$	5 +	4=	9
			20300
$z^4+l^4=$	9	9=	18
			2300
$z^5+l^5=$	7	15=	22
			100
$z^6+l^6=$	3	3	6
			40

And $z=.555973=\log. x$, or $x=3.59728$.

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